CHAPTER 2: ENGINEERING MATHEMATICS

2.1 Introduction

This unit describes the competencies required by a technician in order to apply a wide range of mathematical skills in their work; apply ratios, rates and proportions to solve problems; estimate, carry out measurement; collect, organize and interpret statistical data; use common formulae and algebraic expressions to solve problems.

2.2 Performance Standard

The trainee will apply algebra, trigonometry and hyperbolic functions, complex numbers, coordinate geometry, carryout binomial expansion, calculus ordinary differential equations, Laplace transforms, power series, statistics Fourier series, vector theory, matrix and numerical methods in solving engineering problems.

2.3 Learning Outcomes

2.3.1 List of learning outcomes

a) Apply algebra
b) Apply trigonometry and hyperbolic functions
c) Apply complex numbers
d) Apply coordinate geometry
e) Carry out binomial expansion
f) Apply calculus
g) Solve ordinary differential equations
h) Carry out mensuration
i) Apply power series
j) Apply statistics
k) Apply numerical methods
l) Apply vector theory
m) Apply matrix

2.3.2 Learning Outcome No 1: Apply Algebra
2.3.2.1 Learning Activities

Learning Outcome No 1: Apply Algebra	
Learning Activities	**Special Instructions**
1.1 Perform calculations involving Indices 1.2 Perform calculations involving Logarithms 1.3 Use scientific calculator to solve mathematical problems 1.4 Perform simultaneous equations 1.5 Calculate quadratic equations	• Group assignments

2.3.2.2 Information Sheet No2/LO 1: Apply Algebra

Introduction

This learning outcome covers algebra and the learner should be able to: perform calculations involving Indices as per the concept; perform calculations involving Logarithms as per the concept; use scientific calculator is used mathematical problems in line with manufacturer's manual; perform simultaneous equations as per the rules. Algebra is used throughout engineering, but it is most commonly used in mechanical, electrical, and civil branches due to the variety of obstacles they face. Engineers need to find dimensions, slopes, and ways to efficiently create any structure or object.

Definition of key terms

Algebra is the study of mathematical symbols and the rules for manipulating these symbols; it is a unifying thread of almost all of mathematics. It includes everything from elementary equation solving to the study of abstractions such as groups, rings, and fields.

Content/Procedures/Methods/Illustrations

1.1 Calculations involving Indices are performed as per the concept

Indices

An i**ndex** number is a number which is raised to a power. The power, also known as the index, tells you how many times you have to multiply the number by itself. For example, 2^5 means that you have to multiply 2 by itself five times $= 2 \times 2 \times 2 \times 2 \times 2 = 32$

Laws of indices

(i) $x^0 = 1$

(ii) $x^{-n} = \frac{1}{x^n}$

(iii) $x^n . x^m = x^{n+m}$

(iv) $x^n - x^m = x^{n-m}$

(v) $(x^n)^m = x^{m.n}$

(vi) $x^{\frac{n}{m}} = \sqrt[m]{x^n}$

Application of rules of indices in solving algebraic problems

a) $y^a \text{ x } y^b = y^{a+b}$

Examples:

$2^4 \text{ x } 2^8 = 2^{12}$

$5^4 \text{ x } 5^{-2} = 5^2$

b) $y^a \div y^b = a - b$

Examples

$5^4 \div 5^8 = 5^{-4}$

$7^4 \div 7^{-2} = 7^6$

c) $ym/n = (n\sqrt{y})m$

Examples

$16^{1/2} = \sqrt{16} = 4$

$8^{2/3} = (^3\sqrt{8})^2 = 4$

d) $(y^n)^m = y^{nm}$

Example

$2^5 + 8^4$

$= 25 + (2^3)^4$

$= 2^5 + 2^{12}$

e) $y^0 = 1$

Example

$5^0 = 1$

1.2 Calculations involving Logarithms are performed as per the concept

If **a** is a positive real number other than 1, then the logarithm of **x** with base **a** is defined By:

$$y = \log_a x \quad \text{or } x = a^y$$

Laws of logarithms

(i) $\log_a(xy) = \log_a x + \log_a y$

(ii) $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$

(iii) $\log_a(x^n) = n\log_a x$ for every real number

1.3 Scientific calculator is used in solving mathematical problems

Use the scientific calculator manufacturer's manual on the steps to be followed in doing so.

1.4 Simultaneous equations are performed as per the rules

Simultaneous equations are equations which have to be solved concurrently to find the unique values of the unknown quantities which are true for each of the equations. Two common methods of solving simultaneous equations analytically are:

(i) By substitution

(ii) By elimination

Simultaneous equations with three unknowns

Examples

Solve the following simultaneous equation by substitution methods

$3x - 2y + = 1 \ldots\ldots\ldots..(i)$

$x - 3y + 2z = 13 \ldots\ldots.(ii)$

$4x - 2y + 3z = 17 \ldots\ldots\ldots(iii)$

From equation (ii) $x = 13 + 3y - 2z$

Substituting these expression $(13 + 3y - 2z$ for x gives$)$

$3(13 + 3y - \quad 2z) + 2y + z = 1$

$39 + 9y - 6z + 2y + x = 1$

$11y - 5z = -38 \ldots\ldots\ldots\ldots\ldots..(iv)$

$4(13 + 3y - 2z) + 3z - 2y = 17$

$52 + 12y - 8z + 3z - 2y = 17$

$10y - 5z = -35 \ldots..(v)$

Solve equation (iv) and (v) in the usual way,

From equations (iv) $5z = 11y + 38; z = \frac{11y+38}{5}$

Substituting this in equation (v) gives:

$10y - 5\left(\frac{11y+38}{5}\right) = -35$

$10y - 11y - 38 = -35$

$-y = -35 + 38 = 3$

$y = -3$

$$z = \frac{11y + 38}{5} = \frac{-33 + 38}{5} = \frac{5}{5} = 1$$

But $x = 13 + 3y - 2z$

$x = 13 + 3(-3) - 2(1)$

$=13 - 9 - 2$

$=2$

Therefore, $x = 2$, $y = -3$ and $z = 1$ is the required solution

For more worked examples on substitution and elimination method refer to Engineering Mathematics by A.K Stroud.

1.5 Quadratic equations are calculated as per the concept

Quadratic Equations

Quadratic equation is one in which the highest power of the unknown quantity is 2. For example, $2x^2 - 3x - 5 = 0$ is a quadratic equation.

The general form of a quadratic equation is $ax^2 + bx + c = 0$, where a, b and c are constants and $a \neq 0$ of solving quadratic equations.

1) By factorization (where possible)
2) By completing the square
3) By using quadratic formula
4) Graphically

Example

Solve the quadratic equation $x^2 - 4x + 4 = 0$ by factorization method

Solution

$x^2 - 4x + 4 = 0$

$x^2 - 2x - 2x + 4 = 0$

$x(x - 2) - 2(x - 2) = 0$

$(x - 2)(x - 2) = 0$

i.e. $x - 2 = 0$ or $x - 2 = 0$

$x = 2$ or $x = 2$

I.e. the solution is $x = 2$ (twice)

For more worked examples on how to solve quadratic equations using, factorization, completing the square, quadratic formula refers to basic engineering mathematics by J.O Bird, Engineering mathematical by K.A strand, etc.

Conclusion

The learning outcome covered or equipped the learner with knowledge, skills and attitude to perform calculations involving Indices as per the concept; perform calculations involving Logarithms as per the concept; use scientific calculator in mathematical problems in line with manufacturer's manual; perform simultaneous equations as per the rules.

Further Reading

1. Stroud, A.K. (n.d.). Engineering Mathematics

2.3.2.3 Self-Assessment

Written Assessment

1. Solve the following by factorization
   a) $x^2 + 8x + 7 = 0$
   b) $x^2 - 2x + 1 = 0$
2. Solve by completing the square the following quadratic equations
   a) $2x^2 + 3x - 6$
   b) $3x^2 - x - 6 = 0$
3. Simplify as far as possible
   a) $\log(x^2 + 4x - 3) - \log(x + 1)$
   b) $2\log(x - 1) - \log(x^2 - 1)$
4. Solve the following simultaneous equations by the method of substitution
   a) $x + 3y - z = 2$
   b) $2x - 2y + 2z = 2$
   c) $4x - 3y + 5z = 5$

5. Simplify the following

$$F = (2_{x}^{\frac{1}{2}} y^{\frac{1}{4}})^4 \div \sqrt{\frac{1}{9}x^2y^6} \text{ x } (4\sqrt{x^2y^4})^{-1/2}$$

Oral Assessment

What is your understanding of algebra?

2.3.2.4 Tools, Equipment, Supplies and Materials

- Scientific Calculators
- Rulers, pencils, erasers
- Charts with presentations of data
- Graph books
- Dice
- Computers with internet connection

2.3.2.5 References

Khuri, A. I. (2003). Advanced calculus with applications in statistics (No. 04; QA303. 2, K4 2003.). Hoboken, NJ: Wiley-Interscience.

Stoer, J., &Bulirsch, R. (2013). Introduction to numerical analysis (Vol. 12). Springer Science & Business Media.

Zill, D., Wright, W. S., & Cullen, M. R. (2011). Advanced engineering mathematics. Jones & Bartlett Learning.

2.3.3 Learning Outcome No 2: Apply Trigonometry and Hyperbolic Functions
2.3.3.1 Learning Activities

Learning Outcome No 2: Apply Trigonometry and Hyperbolic Functions	
Learning Activities	**Special Instructions**
2.1 Perform calculations using trigonometric rules 2.2 Perform calculations using hyperbolic functions	

2.3.3.2 Information Sheet No2/LO2: Apply Trigonometry and Hyperbolic Functions

Introduction

This learning outcome equips the learner with knowledge and skills to perform calculations using trigonometric rules and hyperbolic functions.

Definition of key terms

Trigonometry: This is a branch of mathematics which deals with the measurement of sides and angles of triangles and their relationship with each other. Two common units used for measuring angles are degrees and radians.

Content/Procedures/Methods/Illustrations

2.1 Calculations are performed using trigonometric rules

Trigonometric ratios

The three trigonometric ratios derived from a right-angled triangle are the sine, cosine and tangent functions. Refer to basic engineering mathematics by J.0 Bird to read move about trigonometry ratios.

Solution for right angled triangles

To solve a triangle means to find the unknown sides and angles; this is achieved by using the theorem of Pythagoras and or using trigonometric ratios.

Example

Express $3 \text{ Sin } \theta + 4 \text{ Cos}\theta$ in the general form $R \text{ Sin}(\theta + \alpha)$

Let $3 \text{ Sin } \theta + 4 \text{ Cos } \theta = R \text{ Sin } (\theta + \alpha)$

Expanding the right-hand side using the compound angle formulae gives

$3 \text{ Sin } \theta + 4 \text{ Cos } \theta = R\ [\text{Sin}\theta \text{ Cos}\alpha + \text{Cos } \theta \text{ Sin}\alpha\ ]$

$= R \cos\alpha \text{ Sin } \theta + R \text{ Sin } \alpha \text{ Cos } \theta$

Equating the coefficient of:

$\text{Cos}\theta: \ = \ \text{R Sin}\ \alpha \ \text{i.e Sin}\ \alpha = \dfrac{4}{\text{R}}$

$\text{Sin}\ \theta: \quad 3 \ = \ \text{R Cos}\ \alpha \ \ \text{i.e}\ \ \text{Cos}\ \alpha = \frac{3}{\text{R}}$

These values of R and α can be evaluated.

$\text{R} = \sqrt{4^2 + 3^2 = 5}$

$\alpha = \tan^{-1}\dfrac{4}{3} = 53.13^0 \text{or}\ \ 233.13^0$

Since both Sin α and Cos α are positive, r lies in the first quadrant where all are positive, hence 233.13^0 is neglected.

Hence

$3\ \text{Sin}\theta + 4\ \text{Cos}\ \theta = 5\ \text{Sin}\ (\theta + 53.13^0)$

Example

Solve the equation $3\ \text{Sin}\ \theta + 4\ \text{Cos}\ \theta = 2$ for values of θ between 0^0 and 360^0 inclusive

Solution

From the example above

$3\ \text{Sin}\theta + 4\ \text{Cos}\ \theta = 5\ \text{Sin}\ (\theta + 53.13^0)$

Thus

$5\ \text{Sin}(\alpha + 53.13^0) = 2$

$\text{Sin}(\theta + 53.13^0) = \dfrac{2}{5}$

$\theta + \ 53.13^0 = \text{Sin}^{-1}\ 2/5$

$\theta + \ 53.13^0 \ = \ 23.58^0 \text{or}\ 156.42^0$

$\theta = \ 23.58^0 - \ 53.13^0 = -29.55^0$

$=330.45^0$

OR $\theta = \ 156.42^0 - \ 53.13^0$

$=103.29^0$

Therefore, the roots of the above equation are 103.29^0 or 330.45^0

For more worked examples refer to Technician mathematics book 3 by J.) Bird.

Double/multiple angles

For double and multiple angles refer to Technician mathematics by J.O Bird

Factor Formulae

For worked exampled refer to Technic mathematics book 3 by J. O Bird, Pure mathematics by backhouse and Engineering mathematics by KA Stroud.

Half-angle formulae

Refer to pure mathematics by backhouse and Engineering mathematics by K.A STROUD

2.2 Calculations are performed using hyperbolic functions

Hyperbolic functions

Definition of hyperbolic functions, Sinhx cosh x and tanh x

- Evaluation of hyperbolic functions
- |Hyperbolic identifies
- Osborne's Rule
- Solve hyperbolic equations of the form a $\cosh x + b\sinh x = C$

For all the above refer, to engineering mathematics by KA strand.

Conclusion

The learning outcome covered or equipped the learner with knowledge, skills and attitude to perform calculations using trigonometric rules and perform calculations using hyperbolic functions.

Further Reading

1. Stoer, J., &Bulirsch, R. (2013). Introduction to numerical analysis (Vol. 12). Springer Science & Business Media.

2.3.3.3 Self-Assessment

Written Assessment

1. A surveyor measures the angle of elevation of the top of a perpendicular building as19^0. He moves 120m nearer the building and measures the angle of elevation as47^0. Calculate the height of the building to the nearest meter.

2. Solve the equation $5\cos\theta + 4\,\mathrm{Sin}\theta = 3$ for values of θ between 0^0 and 360^0 Inclusive.
3. Prove their identifies
    a) $\mathrm{Cash}\ 2x = \mathrm{cash}^2 x + \mathrm{Sinh}^2 x$
    b) $\mathrm{Sinh}(x + y)\mathrm{SinhCoshy}\ + \mathrm{coshySinhx}$
4. Solve the equation
5. $3\ \mathrm{Sinhx} + 4\ \mathrm{Coshx} = 5$

2.3.3.4 Tools, Equipment, Supplies and Materials

- Scientific calculators
- Rulers, pencils, erasers
- Charts with presentations of data
- Graph books
- Dice
- Computers with internet connection

2.3.3.5 References

Khuri, A. I. (2003). Advanced calculus with applications in statistics (No. 04; QA303. 2, K4 2003.). Hoboken, NJ: Wiley-Interscience.

O'Neil, P. V. (2011). Advanced engineering mathematics. Cengage learning.

Stoer, J., &Bulirsch, R. (2013). Introduction to numerical analysis (Vol. 12). Springer Science & Business Media.

2.3.4 Learning Outcome No 3: Apply Complex Numbers

2.3.4.1 Learning Activities

Learning Outcome No3: Apply Complex Numbers	
Learning Activities	**Special Instructions**
3.1 Represent complex numbers are represented 3.2 Perform operations involving complex numbers are performed 3.3 Perform calculations involving complex numbers	

2.3.4.2 Information Sheet No2/LO3: Apply complex numbers

Introduction

This learning outcome covers an introduction to complex numbers, their representation in argand diagrams and calculations involving complex numbers using De Moivre's theorem

Definition of key terms

Argand diagram: A diagram on which complex numbers are represented geometrically using Cartesian axes; the horizontal coordinate representing the real part of the number and the vertical coordinate representing the imaginary part of the number, as depicted below

Content/Procedures/Methods/Illustration

A number of the form $a + ib$ is called complex number where a and b are real numbers and $i = \sqrt{-1}$ we call $'a'$ the real part and 'b' the imaginary part of the complex $a + ib$ if $a = o$ then ib is said to be purely imaginary, if $b = 0$ the number is real.
Pair of complex number $a + ib$ are said to be conjugate of each other.

Addition and subtraction of complex numbers

Addition and subtraction of complex numbers is achieved by adding or subtracting the real parts and the imaginary parts.

Example 1

$(4 + j5) + (3 - j2)$

$(4 + j5) + (3 - j2)=4 + j5 + 3 - j2$

$=(4 + 3) + j(5 - 2)$

$=7 + j3$

Example 2

$(4 + j7) - (2 - js) = 4 + j7 - 2 + js = (4 - 2) + j(7 + 5)$

$= 2 \ + j12$

Multiplication of complex numbers

Example 1

$(3 + j4)(2 + j5)$

$6 + j8 + j15 + j^2 20$

$6 + j23 - 20 \text{ (since} j^2 = -1)$

$= -14 + j^{23}$

Examples 2

$(5 + j8)(5 - j8)$

$(5 + j8)(5 - j8) = 25 + j40 - j40 - j^2 64$

$= 25 + 64$ =89

A pair of complex numbers are called conjugate complex numbers and the product of two conjugate. Complex numbers is always entirely real.

$\cos\theta + j\sin\theta$

Argand diagram

Although we cannot evaluate a complex number as a real number, we can represent diagrammatically in an argand diagram. Refer to Engineering Mathematics by K.A Stroud to learn more on how to represent complex numbers on an argand diagram. Use the same back learn three forms of expressing a complex number.

Demoivre's Theorem

Demoivre's theorem states that $[r(\cos\theta + j\sin\theta)]^n = \ r^n(\cos n\theta + j\sin n\theta)$

It is used in finding powers and roots of complex numbers in polar

Example

Find the three cube roots of $z = 5(\cos 225^0 + \ j\sin 225^0)$

$$Z_1 = Z^{\frac{1}{3}}\left(\text{Cos}\frac{225^0}{3} + j\sin\frac{225^0}{3}\right)$$

$1.71\ (\cos 75^0 + j\sin 75^0$

$z_1 = 1.71\ (\cos 75^0 + j\text{Sin}75^0)$

Cube roots are the same size (modules) i.e. 1.71 and separated at intervals of $\frac{360^0}{3}$, i.e 120^0

$z_{1=1.71} / 75^0$

$z_{2=1.71} \cos(195^0 + j\mathrm{Sin}195^0$

z_s = 1.71 $(315^+ + j\ \mathrm{Sin}315^0)$

1.1 Complex numbers are represented using Argand diagrams

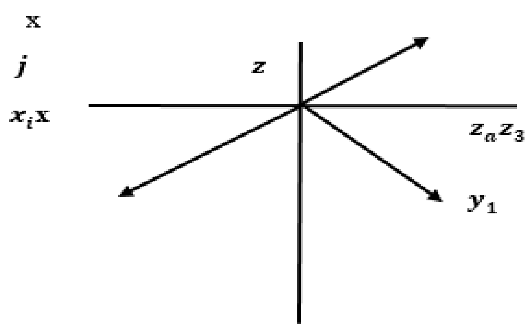


Figure 1: Sketched Argand diagram.

Refer to engineering mathematics by K.A Stroud and learn more on how to find the expansion of

$\cos^n\theta \mathrm{And} \cos^n\theta$

1.2 Operations involving complex numbers are performed

LOCI problems

We sometimes required finding the locus of a point which moves in the Argand diagram according to some stated condition.

Examples

If$Z = x + jy$, find the equation of the locus $\left[\frac{z+1}{z-1}\right] = 2$

$\sin\theta Z = x + jy.$

$\therefore \left(\frac{Z+1}{z-1}\right) = \frac{r_1}{r_2} = \left(\frac{z_1}{z_2}\right) = \frac{[(x+1)^2 + y^2]}{[(x-1)^2) + y^2]}$

$$\frac{[(x+1)^2 + y^2]}{(x-1)^2 + y^2}$$

$$\therefore \frac{(x+1)^2 + y^2}{(x-1)^2 + y^2} + 4$$

$\therefore (x+1)^2 + y^2 = 4\ ((x-1)^2 + y^2)$

$x^2 + 2x + 1 + y^2 = 4(x^2 - 2x + 1 + y^2)$

$=4x^2 - 8x + 4 + 4y^2$

$\therefore 3x^2$-10x+3+$3y^2 = 0$

Conclusion

The learning outcome covered or equipped the learner with knowledge, skills and attitude to represent complex numbers using Argand diagrams, perform operations involving complex numbers and Perform calculations involving complex numbers using De Moivre's theorem.

Further Reading

1. Atkinson, K. E. (2008). An introduction to numerical analysis. John Wiley & Sons.

2.3.4.3 Self-Assessment

Written Assessment

1. Find the fifth roots of $-3 + j3$ in polar form and in exponential form
2. Determine the three cube roots of $\frac{2-j}{2+j}$ giving the results in a modulus/ argument form.
3. Express the principal root in the form $a + jb$
4. If $z = x + jy$, where x and y are real, show that the locus $\left(\frac{Z-2}{z+2}\right) = 2$ is a circle and
5. Determine its center and radius.

Oral Assessment

1. Describe an Argand diagram according to your understanding.
2. What is a complex number?

Practical Assessment

1. Give an example of a complex number. Represent it in an Argand diagram
2. Find the root loci of the complex number above. Use De-Moivre's theorem

2.3.4.4 Tools, Equipment, Supplies and Materials

- Scientific Calculators
- Rulers, pencils, erasers
- Charts with presentations of data
- Graph books
- Dice
- Computers with internet connection

2.3.4.5 References

Atkinson, K. E. (2008). An introduction to numerical analysis. John Wiley & Sons.

Stoer, J., & Bulirsch, R. (2013). Introduction to numerical analysis (Vol. 12). Springer Science & Business Media.

2.3.5 Learning Outcome No 4: Apply Coordinate Geometry

2.3.5.1 Learning Activities

Learning Outcome No 4: Apply Coordinate Geometry	
Learning Activities	**Special Instructions**
4.1 Calculate polar equations 4.2 Draw graphs of given polar equations 4.3 Determine normal and tangents	

2.3.5.2 Information Sheet No2/LO4: Apply Coordinate Geometry

Introduction

This learning outcome covers calculation of polar equations using coordinate geometry, drawing graphs of given polar equations using the Cartesian plane, determining normal and tangents using coordinate geometry.

Content/Procedures/Methods/Illustrations

The position of a point in a plane can be represented in two forms

i) Cartesian co-ordinate (x, y)

ii) Polar co-ordinate (r, θ)

The position of a point in the corresponding axis can therefore generate Cartesian and polar equations which can easily change into required form to fit the required result.

4.1 Polar equations are calculated using coordinate geometry

Example

Convert $r^2 = \sin\theta$ into Cartesian form.

$\cos\theta = \frac{x}{y} \quad \sin\theta = \frac{y}{x}$

Form Pythagoras theorem $r^2 = x^2 + y^2$

$r^2 = \sin\theta$

$(x^2 + y^2) = \frac{y}{x}$

$(x^2 + y^2)x = y$

$(x^2 + y^2)(x^2 + y^2)^{\frac{1}{2}} = y$

$(x^2 + y^2)^{\frac{3}{2}} = y$

Example 2

Find the Cartesian equation of

(i) $r = a(1 + 2\cos)$ (ii) $r\cos(\theta - \alpha) = p$

[The $\text{Cos}\theta$ suggest the relation$X = \text{COS}\theta$, so multiplying through by r}

$\therefore r^2 = a(r + 2r\cos\theta)$

$\therefore x^2 + y^2 = a\left(\sqrt{(x^2 + y^2)} + 2x\right)$

$\therefore x^2 + y^2 + 2x = a\sqrt{(x^2 + y^2)}$

Therefore, the Cartesian equation of $r = a(1 + 2\cos)$ is $(x^2 + y^2 - 2ax)^2 = a^2(x^2 + y^2)$

(ii) $r\cos(\theta - \alpha) = p$

$\cos(\theta - \alpha)$ May be expanded

$\therefore r\cos\theta\cos\alpha + r\sin\theta\sin\alpha = p$

(iii) Therefore, the Cartesian equation of $r\cos(\theta - \alpha) = p$ is $x\cos\alpha + y\sin\alpha = p$

Example 3

Find the polar equation of the circle whose Cartesian equation is $x^2 + y^2 = 4x$

$x^2 + y^2 = 4x$

Put $x = r\cos\theta, y = r\sin\theta$, then

$r^2\cos^2\theta + r^2\sin^2\theta = 4r\cos\theta$

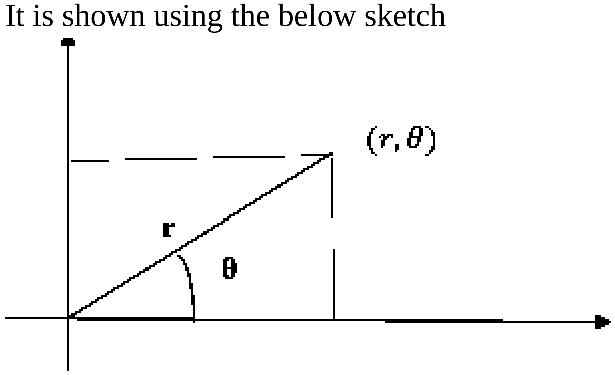
$\therefore r^2 = 4r\cos\theta$

Therefore, the polar equation of the circle is$r^2 = 4r\cos\theta$.

For more information on the conversion of Cartesian equation to polar equation and vice versa refer to pure mathematics by J.K Backhouse.

4.2 Draw graphs of given polar equations using the Cartesian plane

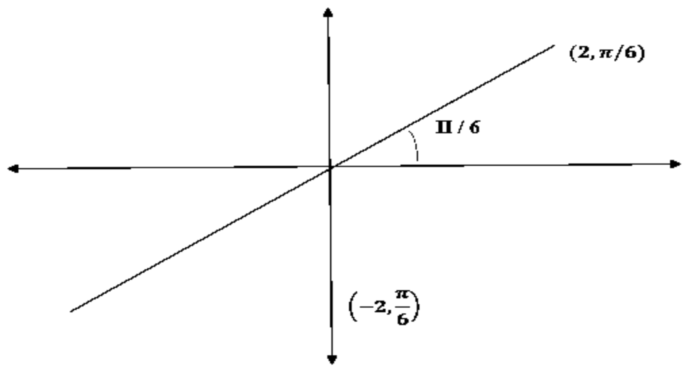
It is shown using the below sketch



Example:

Given points $\left(2, \frac{\pi}{6}\right)$ and $\left(-2, \frac{\pi}{6}\right)$

Sketch polar equation on the Cartesian plane



Example

Change $x = 3t + 1, y = 2t - 5$ to rectangular coordinates and then graph

$x = 3t + 1\ (equation\ 1)$

$y = 2t - 5\ (equation\ 2)$

From equation 1

$T = \frac{x - 1}{3}$

Substitute to equation 2

We have:

$Y = 2((x - 1) / 3) - 5$

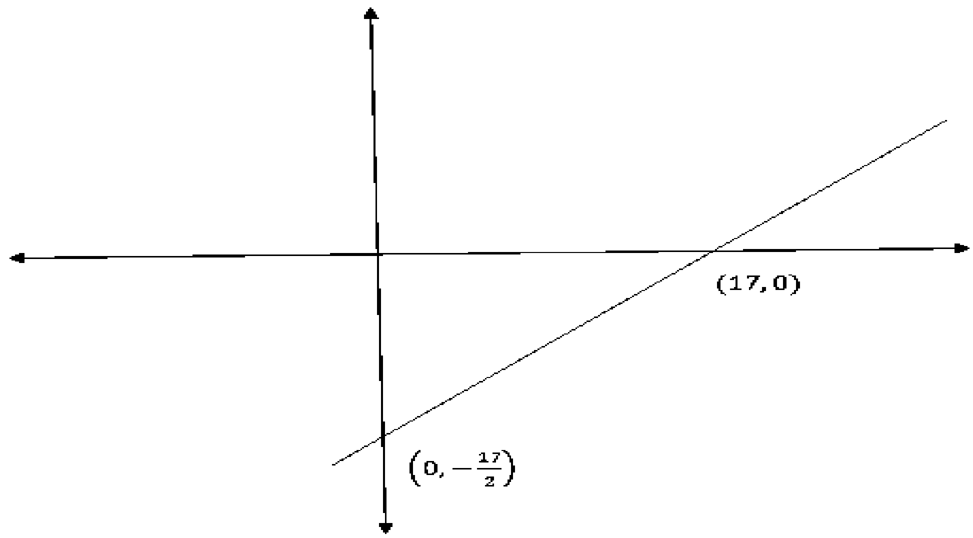
Simplifying

$3y - 2x + 17 = 0$

To graph, simply to intercept from

$\frac{X}{a} + \frac{y}{b} = 1$

$\frac{X}{\frac{17}{2}} + \frac{y}{-\frac{17}{2}}$



Example 1 Source (www Analysemath.com)
Graph the polar equation given by $R = 4 \cos t$ and identify the graph.

Solution

T	R
0	4
π / 6	3.5
π / 4	2.8
π / 3	2
π / 2	0
2 π / 3	-2
3 π / 4	-2.8
5 π / 6	-3.5
Π	4

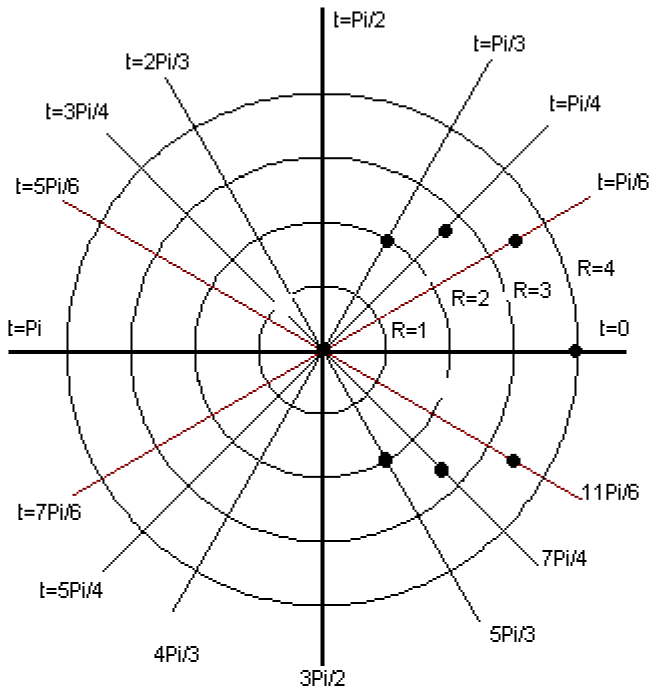
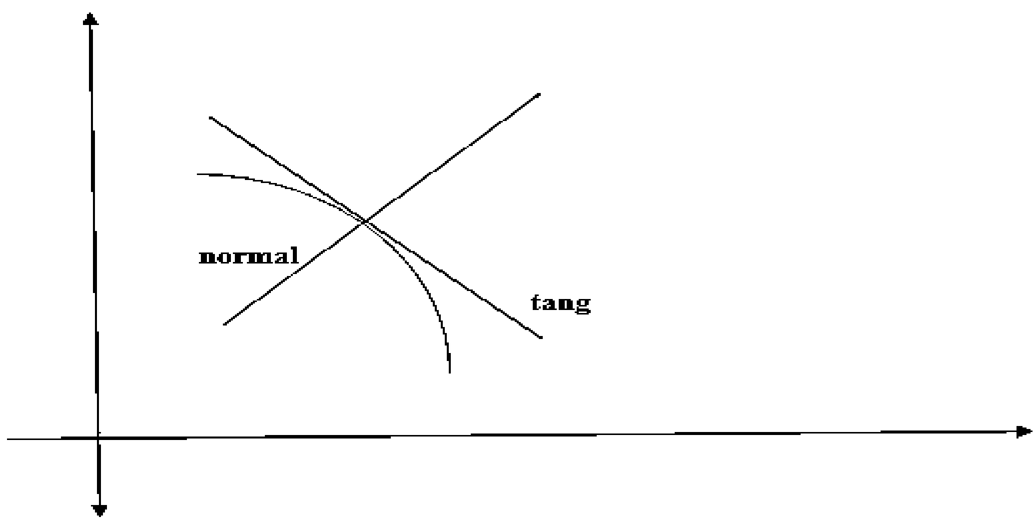


Figure 2: Solution on plotting polar equations.
Source www.analysemath.com

4.3 Determine normal and tangents using coordinate geometry

A tangent is a line that touches the curve. A normal is the perpendicular line to the tangent of the curve.



Note:

The tangent to a curve of any point will be parallel to $X - axis$ if $\theta = 0$ i.e. the derivative at the point will be zero.

i.e. $\left[\frac{dx}{dy}\right]$ at $(x, y) = 0$

The tangent at a point to the curve $y = f(x)$ will be parallel to $Y - axis$ if $\frac{dy}{dx} = 0$ at that point.

Example 1

Find the point on the curve $y = 3x^2 - 2x + 1$ at which the slope of the gradient is 4

$\frac{dy}{dx} = 6x - 2$

$6x - 2 = 4$

$6x = 6$

$x = 1, y = 2$, required point = $(1, 2)$

Example 3

Find the slope of tangent and normal to the curve $x^2 + x^3 + 3xy + y^2 = 5$ at (1, 1)

Solution:

The equation of the curve is

$x^2 + x^3 + 3xy + y^2 = 5$

Differentiating (1) w.r.t, we get

$2x + 3x^2 + 3\left(x\frac{dy}{dx} + y.1\right) + 2y\left(\frac{dy}{dx}\right) = 0$

Substituting$x = 1, y = 1$ in (2) we get

$2 \times 1 + 3 \times 1 + 3\left(\frac{dy}{dx} + 1\right) + 2\frac{dy}{dx} = 0$

∴ The slope of the tangent to the curve at $(1, 1)$ is $-\frac{8}{5}$

∴ The slope of normal to the curve at $(1, 1)$ is $\frac{5}{8}$

Point is a set of values that describes its position on a two- or three-dimensional plane.

Conclusion

The learning outcome covered or equipped the learner with knowledge, skills and attitude to calculate polar equations using coordinate geometry, draw graphs of given polar equations using the Cartesian plane, determine normal and tangents using coordinate geometry.

Further Reading

Kreyszig, E. (1999), Advanced Engineering Mathematics, 8th ed., John Wiley (New York).
O'Neil, P.V. (1995), Advanced Engineering Mathematics, 4th ed., PWS-Kent Pub. (Boston).

2.3.5.3 Self-Assessment

Written Assessment

1. Obtain the polar equation of the following loci
   a) $x^2 + y^2 = a^2$
   b) $x^2 - y^2 = a^2$
   c) $y = 0$
   d) $y^2 = 4a(a - x)$
   e) $x^2 + y^2 - 2y = 0$
   f) $xy = c^2$

2. Obtain the Cartesian equation of the following loci
   a) $r = 2$
   b) $a(1+\cos\theta)$
   c) $r = a\cos\theta$
   d) $r = a\tan\theta$
   e) $r = 2a(1 + \sin 2\theta)$
   f) $2r^2\sin 2\theta = c^2$
   g) $\frac{l}{r} = 1 + 8\cos\theta$
   h) $r = 4a\cot\theta\,\mathrm{cosec}\theta$

2.3.5.4 Tools, Equipment, Supplies and Materials

- Scientific Calculators
- Rulers, pencils, erasers
- Charts with presentations of data
- Graph books
- Dice
- Computers with internet connection

2.3.5.5 References

Greenberg, M.D. (2000), Advanced Engineering Mathematics, 2nd ed., Prentice Hall (Upper Saddle River, N.J).

Hildebrand, F.B. (2002), Introduction to Numerical Analysis, 2nd ed., McGraw-Hill (New York).

Hildebrand, F.B. (2000), Advanced Calculus for Applications, 2nd ed., Prentice-Hall (Englewood Cliffs, NJ).

2.3.6 Learning Outcome No 5: Carry Out Binomial Expansion

2.3.6.1 Learning Activities

Learning Outcome No 5: Carry Out Binomial Expansion	
Learning Activities	**Special Instructions**
5.1 Determine roots of numbers using binomial theorem 5.2 Determine errors of small changes using binomial theorem	

2.3.6.2 Information Sheet No2/LO5: Carry Out Binomial Expansion

Introduction

This learning outcome seeks to equip the learner with knowledge and skills to determine the roots of numbers using binomial theorem and to determine errors of small changes using binomial theorem.

Content/Procedures/Methods/Illustrations

5.1 Carry out Binomial expansion

Binomial is a formula for raising a binomial expansion to any power without lengthy multiplication. It states that the general expansion of $(a+b)^n$ is given as

$$(a+b)^n = a^n b^0 + na^{n-1}n^1 + \frac{n(n-1)a^{n-2}b^2}{2!} + \frac{n(n-1)(n-2)a^{n-3}b^3}{3!} + \ldots$$

Where n can be a fraction, a decimal fraction, positive or negative integer.

Example 1

Use binomial theorem to expand $(2+x)^3$

Solution

$$(a+b)^n = a^n b^0 + na^{n-1}n^1 + \frac{n(n-1)a^{n-2}b^2}{2!} + \frac{n(n-1)(n-2)a^{n-3}b^3}{3!} + \ldots$$

$$A = 2, b = x \text{ and } n = 3$$

$$(2+x)^3 = 2^3x^0 + 3X2^2x^1 + \frac{3(3-1)2^1x^2}{2!} + \frac{3(3-1)(3-2)2^0x^3}{3!} + \ldots$$

$$= 8 + 12x + 6x^2 + x^3$$

For more examples on positive power refer to Technician Mathematic Book by J.O Bird.

Binomial theorem for any index

It has been shown that:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots.$$

The series may be continued indefinitely for any value of n provided $-1 < x < 1$

Example

Use the binomial theorem to expand $\frac{1}{1-x}$ in ascending power of x as far as the term in x^3.

Solution

Since $\frac{1}{1-x}$ may be written$(a - x + x)^{-1}$, the binomial theorem may be used. Thus

$(1-x)^{-1} = 1 + -1(-x) + \frac{-1(-2)}{2!}x^2 + \frac{-1(-2)(-3)}{3!}+\ldots.$

$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$

Provided $-1 < x < 1$

5.2 Practical application of binomial theorem

Example1

The radius of a cylinder is reduced by 4% and its height increased by 2%. Determine the appropriate percentage change in its volume neglecting products of small quantities.

Solution

Volume, $V = \pi r^2 h$

Let original values be, radius $= r$

Height $= h$

New values radius $= (1 - 0.04)r$

Height $= (1 + 0.02)h$

New volume $= \pi(1 - 0.004)^2 r^2 (1 + 0.02)h$

Using binomial theorem, $(1 - 0.04)^2 = 1 - 2(0.04) + (0.04)^2 = 1 - 0.08$

$= \pi r^2 h(1 - 0.08)(1.02) = \pi r^2 h(0.94)$

Percentage change $= \frac{(0.94-1)100\%}{1} = -6\%$

The new volume decreased by 6%

Conclusion

The learning outcome covered or equipped the learner with knowledge, skills and attitude to determine the roots of numbers using binomial theorem and to determine errors of small changes using binomial theorem.

Further Reading

1. Hoyland, A., Rausand, and M. (1994), System Reliability Theory: Models and Statistical Methods, John Wiley (New York).
2. Kaplan, W. (1984), Advanced Calculus, 3rd ed., Addison-Wesley (Cambridge, MA).
3. Kreyszig, E. (1999), Advanced Engineering Mathematics, 8th ed., John Wiley (New York).

2.3.6.3 Self-Assessment

Written Assessment

1. Expand as far as the third term and state the limits to which the expansions are valid.
    a) $\frac{1}{(1+2x)^3}$
    b) $\sqrt{4+x}$
2. Show that if higher powers of x are neglected,

$$\sqrt{\frac{1+x}{1-x}} = 1 + x + \frac{x^2}{2}$$

3. The second moment of area of a rectangular section through its centroid is given by $\frac{bl^3}{12}$. Determine the appropriate change in the second moment of area if b is increased by 3.5% and l is reduced by 2.5%.

2.3.6.4 Tools, Equipment, Supplies and Materials

- Scientific Calculators
- Rulers, pencils, erasers
- Charts with presentations of data
- Graph books
- Dice
- Computers with internet connection

2.3.6.5 References

Greenberg, M.D. (1998), Advanced Engineering Mathematics, 2nd ed., Prentice Hall (Upper Saddle River, N.J).

Hildebrand, F.B. (1974), Introduction to Numerical Analysis, 2nd ed., McGraw-Hill (New York).

Hildebrand, F.B. (1976), Advanced Calculus for Applications, 2nd ed., Prentice-Hall (Englewood Cliffs, NJ).

2.3.7 Learning Outcome No 6: Apply Calculus

2.3.7.1 Learning Activities

Learning Outcome No 6: Apply Calculus	
Learning Activities	**Special Instructions**
6.1 Determine derivatives of functions 6.2 Determine derivatives of hyperbolic functions 6.3 Determine derivatives of inverse trigonometric functions 6.4 Determine rate of change and small change 6.5 Perform calculation involving stationery points of functions of two variables 6.6 Determine integrals of algebraic functions 6.7 Determine integrals of trigonometric functions 6.8 Determine integrals of logarithmic functions 6.9 Determine integrals of hyperbolic and inverse functions	

2.3.7.2 Information Sheet No2/LO6: Apply calculus

Introduction

This learning outcome equips the learner with relevant knowledge, skills and attitude so that they are able to: determine the derivatives of functions using differentiation; determine derivatives of hyperbolic functions using differentiation; determine derivatives of inverse trigonometric functions using differentiation; determine rate of change and small change using differentiation; perform calculation involving stationery points of functions of two variables using differentiation; determine integrals of algebraic functions using integration; determine integrals of trigonometric functions using integration; determine integrals of logarithmic functions using integration; determine integrals of hyperbolic and inverse functions using integration.

Definition of key terms

Calculus: It is a branch of mathematics involving calculations dealing with continuously varying functions. The subject falls into two parts namely differential calculus (differentiation) and integral calculus (integration).

Differentiation: The central problem of the differential calculus is the investigation of the rate of change of a function with respect to changes in the variables on which it depends.

Content/Procedures/Methods/Illustrations

6.1 Differentiation from first principles

To differentiate from first principles means to find $f'(x)$ using the expression.

$f'(x) = \lim_{\delta x \to 0} \left\{\frac{f(x+\delta x)}{\delta x}\right\}$

$\delta x \to 0, \left\{\frac{f(x+8x)-f(x)}{\delta x}\right\}$

$f(x) = x^2$

$f(x+\delta x) = (x+\delta x)^2 = x^2 + 2x\delta x + (\delta x)^2$

$f(x+\delta x) - f(x) = x^2 + 2\,x\delta x + (\delta x)^2 - x^2$

$=2x\delta x + (\delta x)^2$

$\frac{f(x+\delta x)-f(x)}{\delta x} = \frac{2x\delta x\ +(\delta x)^2}{\delta x}$

$=2x + \delta x$

As $\delta x \to 0$, $\frac{f(x+\delta x)-f(x)}{\delta x} \to 2x + 0$

$\therefore f'(x) = \lim_{\delta x \to 0} \left\{\frac{f(x+\delta x)-f(x)}{\delta x}\right\} = 2x$

At $x = 3$, the gradient of the curve i.e $f'(x) = 2(3) = 6$

Hence if $f(x) = x^2$, $f'(x) = 2x$. The gradient at $x = 3$ is 6

6.2 Methods of differentiation

There are several methods used to differentiate different functions which include:

(i) Product Rule
(ii) Quotient Rule
(iii) Chain Rule
(iv) Implicit Rule

Example

Determine ${}^{dy}/_{dx}$ given that

a) $y = x^2 \text{Sin}x$

Solution

From product rule: $uv(x) = u\frac{dv}{dx} + v\frac{du}{dx}$

$u = x^2$ and $v = \text{Sin}x$

$\frac{du}{dx} = 2x$ $\frac{dv}{dx} = \text{Cos}x$

$\therefore \frac{dy}{dx} = x^2(\text{Cos}x) + \text{Sin}x\,(2x)$

$= x^2\text{Cos}\,x + 2x\,\text{Sin}\,x$

b) $y = \frac{x^2+1}{x-3}$

Solution. Using Quotient rule:

$$\frac{u(x)}{v(x)} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$u = x^2+1$ $\qquad\qquad$ $v = x - 3$

$\frac{du}{dx} = 2x$ $\frac{dv}{dx}=1$

$\therefore dy/dx = \frac{(x-3)(2x)-(x^2+1)(1)}{(x-3)^2}$

$$= \frac{2x^2 - 6x - x^2 - 1}{(x-3)^2}$$

$$= \frac{x^2 - 6x - 1}{(x-3)^2}$$

$$= \frac{x^2 - 6x - 1}{x^2 - 6x + 9}$$

For more examples on the cases of application of the other highlighted rates refer to Engineering Mathematics by K Stroud.

6.3 Applications of differentiation

Differentiation can be used to determine velocity and acceleration of a moving body. It can also be applied to determine maximum and minimum values.

Example: A rectangular area is formed using a piece of wire 36cm long. Find the length and breadth of the rectangle if it is to enclose the maximum possible area.

Solution.

Let the dimension a rectangle be x and y

Perimeter of rectangle = $2x + 2y = 36$

i.e.$x + y = 18$……………… (i)

Since it is the maximum area that is required, a formula for the area A must be obtained in terms of one variable only.

Area = $A = xy$

From equation (i), $y = 18 - x$

Hence $A = x(18 - x) = 18x - x^2$

Now that an expression for the area has been obtained in terms of one variable it can be differentiated with respect to that variable

$\frac{dA}{dx} = 18 - 2x$ for maximum or minimum value i.e. $x = 9$

$\frac{d^2A}{dx^2} = -2$, which is negative giving a maximum value

$y = 18 - x = 18 - 9 = 9$

Hence the length and breadth of the rectangle for maximum area are both 9 cm i.e. a square gives the maximum possible area for a given perimeter length When perimeter is 36cm, maximum area possible is 81cm^2.

6.4 Determine rate of change and small change using differentiation

We can always write the rate of change of one variable in terms of the other when the two are varying with respect to time. Both the functions will be differentiated with time. We therefore define $dx/_{dt}$ for the same function$f(x)$.

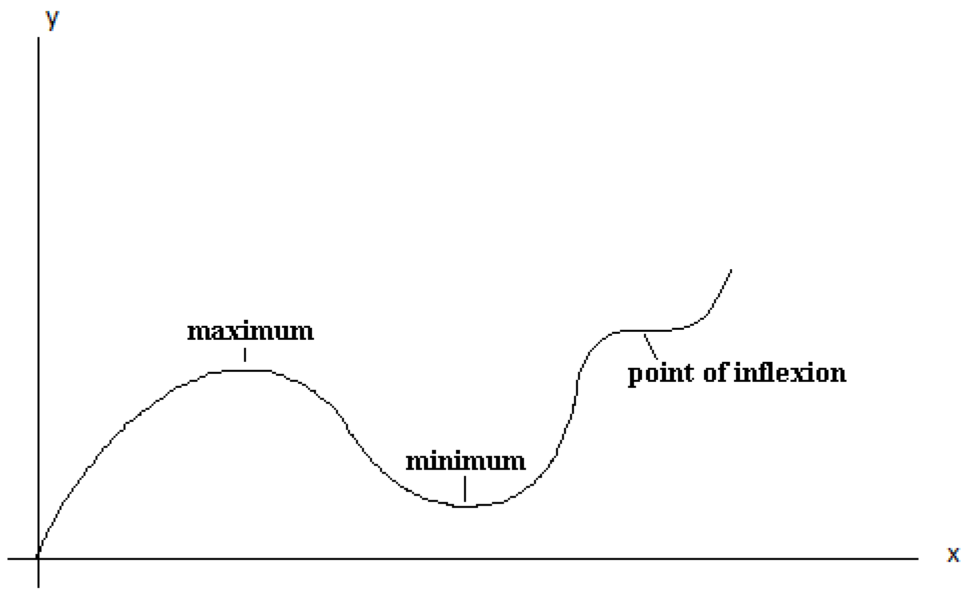
The solution of this problem is done using the following algorithm:

i. Sketch the problem
ii. Identify the constants and variables in the equation
iii. Identify independent and dependent variables
iv. Differentiate with respect to time.
v. Evaluate at a given point.

6.5 Perform calculation involving stationery points of functions of two variables using differentiation

A stationary point k is a point x is one at which$f'(x) = 0$. A point of a function $f(x)$ K is said to be maximum point if and only if $f(k) \geq f9x)$ for all the x in the f domain.
A point k of a function fx said to be a minimum point if and only if for all
x in the f domain, $f(k) \leq f(x)$.

Local Maximum and minimum



The occurrence of local maximum or minimum is determined by the derivations of a given function.

Example

Find the turning points of $f(x) = 3x4 + 16x3 + 24x2$ and hence determine their nature.

Solution

$f(x) = 3x^4 + 16x^3 + 24x^2 + b$

$f'(x) = 12x^2 + 48x$

$= 12(x^3 + 4x^2 + 4x)$

$= 12x(x+2)^2$

Solving the equation $f'(x) = 12x(x+2)^2 = 0$

We get $x = 0 \; or \; x = -2$

$= x < -2 \;, f'(x) < 0$

$-2 < x < 0, \; f'(x) < 0$

$x > 0, \; f'(x) > 0$

We can deduce that at $x = 0 \; and \; x = -2,$ we get stationary point.

Also, at $x = 0$ we have a minimum point and at $x = -2,$ is a point of inflexion.

6.6 Integrals of algebraic functions are determined using integration

Integration

Process of integration reverses the process of differentiation. In differentiation if $f(x) = x^2$, then $f'(x) = 2x$.

Since integration reverse the process of moving from f(x) to $f'(x)$, it follows that the integral of $2x$ is x^2 i.e it is the process of moving from $f'(x)$ to $f(x)$. Similarly if $y = x^3$ then $\frac{dy}{dx} = 3x^2$. Reversing this process shows that the integral of $3x^2$ is x^3.

Integration is also the process of summation or adding parts together and an elongated $'s'$ shown as $\int$ is used to replace the words 'integrated of'. Thus $\int 2x = x^2$ and $\int 3x^2 = x^3$

Refer to Engineering Mathematics by K.A Strand and learn those on definite and indefinite integrals.

6.7 Methods of integration and application of integration

The methods available are:

a) By using algebraic substitution
b) Using trigonometric identities and substitutions
c) Using partial fraction
d) Using $t = \tan\frac{\theta}{2}$ substitution
e) Using integration by parts

Refer to Engineering Mathematics by K. A. Strand and learn more about methods of integration,

Also use the above stated book to learn more on application of integration to find areas, volumes of revolutions, etc.

6.8 Determine integrals of logarithmic functions using integration

The natural logarithm function is always differentiable throughout is (0, 00) domain and

$ln(x) = \frac{1}{x}$ Also, in x increase $0, \infty)$

The rules of integrating logarithms functions are as follows:

f(x)	$\int \boldsymbol{fx(dx)}$
$ln(x)$	$xlnx - x + c$
$\log x$	$(xlnk - x)\ln(10) + c$
$log_a x$	$x(\log ax - \log\ ae) + c$

EXAMPLE:

Solve $\int e^{x2} 2x^2\, dx$

Solution

i. $f(x) = \int e^{x2} 2x^2\, dx$

ii. let $u = x^3$ and $du = 3x^2$

iii. The new function is now: $\int e^{x2} 3x^2 du = \int e^{u}$du

iv. According to laws of integration: $\int e^{x} dx = e^{x} + c$ thus $\int e^{u} du = e^{u} + c$

v. But $u = x^3$ *thus we have* $\int e^{u} du = \int e^{x3} dx = e^{x3} + c$

The answer is hence: $e^{x3} + C$

6.9 Determine integrals of hyperbolic and inverse functions using integration

There are six basic hyperbolic which are defined in section 6.2 above. The table below shows the integrals of hyperbolic function:

Function	**Integral**
$\sinh x$	$cosch\ x + c$
$cash\ x$	$\sinh x + c$
$\tanh x$	$ln/cosh/{+}C$
$\operatorname{csch} x$	$ln/\tanh(\frac{x}{2})/{+}C$
$\operatorname{sech} x$	$\arctan(sinhx) + C = tan^{-1}(sinhx) + c)$
$cotch\ x$	$ln/sinhx + C$

The indefinite integral of one-to-one function is expresses as

$\int f^{-1}(x)dx = xf^{-1}(x) - \int f(u)du\,; u = f^{-1}(x)$

A definite integral is expressed as;

$\int_a^b f^{-1}(x) = bf^{-1}(b) - af^{-1}(a) - \int_{fy(a)}^{f-1(b)} f(u)du.$

Conclusion

The learning outcome covered or equipped the learner with knowledge, skills and attitude to determine the derivatives of functions using differentiation, determine derivatives of hyperbolic functions using differentiation, determine derivatives of inverse trigonometric functions using differentiation, determine the rate of change and small change using differentiation; perform calculation involving stationery points of functions of two variables using differentiation; determine integrals of algebraic functions using integration; determine integrals of trigonometric functions using integration; determine integrals of logarithmic functions using integration; determine integrals of hyperbolic and inverse functions using integration.

Further Reading

Read more on methods of integration.

2.3.7.3 Self-Assessment

Written Assessment

1. Find the co-coordinator, of the points on the curve

   $y = \frac{1/3(5-6x)}{3x^2+2}$

   Where the gradient is zero

2. If $y = \frac{4}{3x^3} - \frac{2}{x^2} + \frac{1}{3x} - \sqrt{x}$. Find $\frac{d^2y}{dx}$ and $\frac{d^3y}{dx}$

3. Find $\int \cos 6x \text{ Sin } 2x \, dx$

4. Evaluate $\int_3^4 \frac{x^3-x^2-5x}{x^2-2x+2} dx$

Oral Assessment

1. What do you understand by differentiation and integration as applied in calculus?
2. How are they intertwined?
3. What are some of their practical applications?

2.3.7.4 Tools, Equipment, Supplies and Materials

- Scientific calculators
- Rulers, pencils, erasers
- Charts with presentations of data
- Graph books
- Computers with internet connection

2.3.7.5 References

Greenberg, M.D. (1998), Advanced Engineering Mathematics, 2nd ed., Prentice Hall (Upper Saddle River, N.J).

Hildebrand, F.B. (1974), Introduction to Numerical Analysis, 2nd ed., McGraw-Hill (New York).

Hildebrand, F.B. (1976), Advanced Calculus for Applications, 2nd ed., Prentice-Hall (Englewood Cliffs, NJ).

2.3.8 Learning Outcome No 7: Solve Ordinary Differential Equations

2.3.8.1 Learning Activities

Learning Outcome No 7: Solve Ordinary Differential Equations	
Learning Activities	**Special Instructions**
7.1. Solve first order and second order differential equations 7.2. Solve first order and second order differential equations from given boundary conditions	

2.3.8.2 Information Sheet No2/LO7: Solve Ordinary Differential Equations

Introduction

This learning outcome equips the learner with knowledge and skills to solve first order differential equations using the method of undetermined coefficients and also when given boundary conditions.

Content/Procedures/Methods/Illustrations

7.1. Solve first order and second order differential equations using the method of undetermined coefficients

An equation involves differential co-efficient is called a differential equation.

Examples

(i) $\frac{dy}{dx} = \frac{1+x^2}{1-y^2}$

(ii) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 8y = 0$

The order of a differential equation is the order of the highest differential coefficient present in the equation. Differential equations represent dynamic relationships i.e. quantities that change, and are thus frequently occurring in scientific and engineering problems.

Formation of a differential equation

Differential equations may be formed in practice from a consideration of the physical problem to which they refer. Mathematically, they can occur when arbitrary constants are eliminated from a given function.

Example

Consider $y = A\sin x + B\cos x$, where A and B are two arbitrary constants. If we differentiate, we get

$\frac{dy}{dx} = A\text{Cos}x - B\text{Sin}x$ and $\frac{d^2y}{dx^2} = -A\text{Sin}x - B\text{Cos}x = -(A\text{Sin}x + B\text{Cos}x)$

i.e. $\frac{d^2y}{dx^2} = -y$

$$\therefore \qquad \frac{d^2y}{dx^2} - y = 0$$

This is a differential equation of the second order.

Types of first order differential equations

a) By separating the variables
b) Homogeneous first order differential equations
c) Linear differential equations
d) Exact differential equations

Application of first order differential equations

Differential equations of the first order have many applications in Engineering and Science.

Example

The rate at which a body cools is given by the equations $\frac{d\theta}{dt} = -k\theta$ where θ the temperature of the body above the surroundings is and k is a constant. Solve the equation for θ given that $t = 0$,

$\theta = \theta_0$

Solution

$$\frac{d\theta}{dt} = -k\theta$$

Rearranging gives: $dt = \frac{-1}{k\theta}$

Integrating both sides gives: $\int dt = \frac{-1}{k}\int \frac{d\theta}{\theta}$

i.e.$t = \frac{-1}{k}\ln\theta + c$………………….. (i)

Substituting the boundary conditions $t = 0$, $\theta = \theta_0$ to find c gives

$$0 = \frac{-1}{k}\ln\theta_0 + c$$

$$\text{i.e. } c = \frac{1}{k}\ln\theta_0$$

Substituting $c = \frac{-1}{k}\ln\theta_0$ in equation (i) gives

$t = \frac{-1}{k}\ln\theta + \frac{1}{k}\ln\theta_0$

$t = \frac{1}{k}(\ln\theta_0 + \ln\theta) = \frac{1}{k}\ln\left(\frac{\theta_0}{\theta}\right)$

$kt = \ln\left(\frac{\theta_0}{\theta}\right)$

$e^{kt} = \frac{\theta_0}{\theta}$

$e^{-kt} = \frac{\theta}{\theta_0}$

Hence, $\theta = \theta_0 e^{-kt}$

7.2 First order and second order differential equations are solved using the method of undetermined coefficients

Formation of the second order differential equation

For formation of second order differential equations refer to Engineering Mathematics by K.A Strand, Technician 4 and 5 by J.O Bird.

Application of second order differential equations

Many applications in engineering give rise to the second order differential equations of the form $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$

Where a, b, c are constant coefficients and $f(x)$ is a given function of x.

Examples include:

(1) Bending of beams
(2) Vertical oscillations and displacements
(3) Damped forced vibrations

Conclusion

The learning outcome covered or equipped the learner with knowledge, skills and attitude to solve first order differential equations using the method of undetermined coefficients; and also when given boundary conditions.

2.3.8.3 Self-Assessment

Written Assessment

1. Solve the following equations:
    a) $x(y-3)\frac{dy}{dx}4y$
    b) $(xy + y^2) + (x^2 - xy)\frac{dy}{dx} = 0$
    c) $\frac{dy}{dx} + y\tan x = \sin x$

2. Show that the change, q, on a capacitor in an LCR circuit satisfies the second order differential equation

$$L\frac{d^2q}{dt^2} + b\frac{dq}{dt} + \frac{1}{c}q = E$$

3. Show that if $2L = cR^2$ the general solution of this equation is

$$q = e^{\frac{-t}{cR}}\left(A\cos\frac{1}{cR}t + B\sin\frac{1}{cR}t\right) + cE$$

4. If $i = \frac{dq}{dt} = 0$ and $q = 0$ when $t = 0$, show that the current in the circuit is

$$i = \frac{2E}{R}e^{\frac{-t}{cR}}\sin\frac{1}{cR}$$

Oral Assessment

1. Consider the following differential equation

   $x(y-3)\frac{dy}{dx} = 4y.$ Is it of the first order or the second order?

2. Justify your answer above

2.3.8.4 Tools, Equipment, Supplies and Materials

- Scientific Calculators
- Rulers, pencils, erasers
- Charts with presentations of data
- Graph books
- Dice
- Computers with internet connection

2.3.8.5 References

Greenberg, M.D. (1998), Advanced Engineering Mathematics, 2nd ed., Prentice Hall (Upper Saddle River, N.J).

Hildebrand, F.B. (1974), Introduction to Numerical Analysis, 2nd ed., McGraw-Hill (New York).

Hildebrand, F.B. (1976), Advanced Calculus for Applications, 2nd ed., Prentice-Hall (Englewood Cliffs, NJ).

2.3.9 Learning Outcome No 8: Carry Out Mensuration

2.3.9.1 Learning Activities

Learning Outcome No 8:Carry Out Mensuration	
Learning Activities	**Special Instructions**
8.1. Obtain perimeter and areas of figures 8.2. Obtain volume and surface area of solids 8.3. Obtain area of irregular figures 8.4. Obtain areas and volumes using Pappus theorem	

2.3.2.9.2 Information Sheet No2/LO8: Carry Out Mensuration

Introduction

This learning outcome covers perimeter and areas of figures, volume and surface area of solids, area of irregular figures and areas and volumes using Pappus theorem**.**

Definition of key terms

Area: Extent of part of a surface enclosed within a boundary.

Circumference: Distance around a circle.

Dimension: Measurable extent such as length, thickness and width.

Fraction: Number expressed as a quotient of two other numbers.

Mensuration: Act or art of measuring.

Perimeter: Bounding line or curve of a plain area.

Standard: Serves as a measure of reference.

Content/Procedures/Methods/Illustrations

8.1 Obtain perimeter and areas of figures

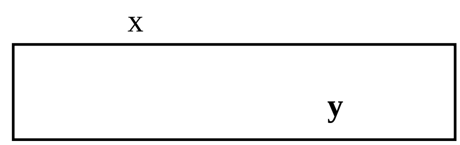
Perimeter

The perimeter is the length of the outline of a shape. To find the perimeter of a rectangle or square you have to add the lengths of all the four sides. x is in this case the length of the rectangle while y is the width of the rectangle.

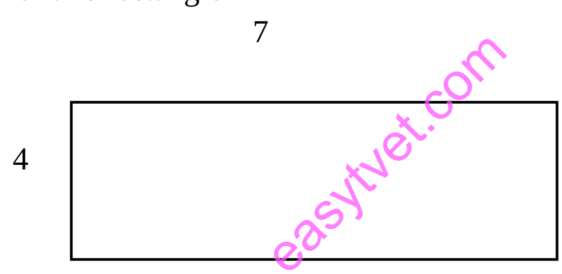
The perimeter, P, is:

$$P = x + x + y + y$$
$$P = 2x + 2y$$

$P=2(x+y)$



Find the perimeter of this rectangle



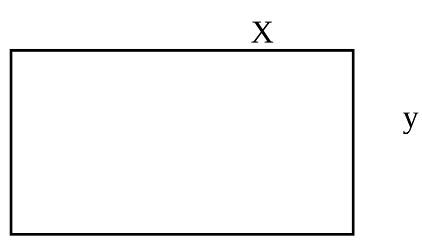
$$P = 7 + 7 + 4 + 4$$
$$P = 2 \cdot 7 + 2 \cdot 4$$
$$P = 2 \cdot (7 + 4)$$
$$P = 2 \cdot 11$$
$$P = 22\text{in}$$

Area

Area is the measurement of the surface of a shape. To find the area of a rectangle or a square you need to multiply the length and the width of a rectangle or a square.

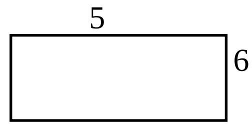
Area, A, is x times y.

$A=x\cdot y$



Examples

Find the area of this square.



$$A = x \cdot y$$
$$A = 5 \cdot 6$$
$$A = 30\text{in}^2$$

There are different units for perimeter and area. Perimeter has the same units as the length of the sides of rectangle or square whereas the area's unit is squared.

8.2 Obtain volume and surface area of solids

The surface area of a figure is defined as the sum of the areas of the exposed sides of an object.

The volume of an object is the amount of three-dimensional space an object takes up. It can be thought of as the number of cubes that are one unit by one unit by one unit that it takes to fill up an object.

Surface Area of a Rectangular Solid (Box)

$$SA = 2(lw + lh + wh)$$
$$l = \text{length of the base of the solid}$$
$$w = \text{width of the base of the solid}$$
$$h = \text{height of the solid}$$

Volume

Volume of a Solid with a Matching Base and Top

V=Ah

A = area of the base of the solid

h = height of the solid

Volume of a Rectangular Solid (specific type of solid with matching base and top)

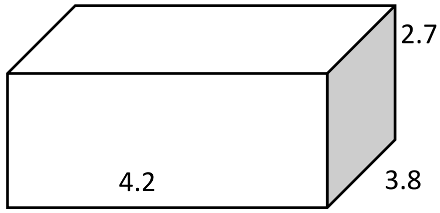
V=lwh

l = length of the base of the solid

w = width of the base of the solid

h = height of the solid

Examples



This figure is a box (officially called a rectangular prism). We are given the lengths of each of the length, width, and height of the box, thus we only need to plug into the formula. Based on the way our box is sitting, we can say that the length of the base is 4.2 m; the width of the base is 3.8 m; and the height of the solid is 2.7 m. Thus we can quickly find the volume of the box to be

$$V = lwh = 4.2 * 3.8 * 2.7 = 43.092m3$$

Although there is a formula that we can use to find the surface area of this box, you should notice that each of the six faces (outside surfaces) of the box is a rectangle. Thus, the surface area is the sum of the areas of each of these surfaces, and each of these areas is fairly straight-forward to calculate. We will use the formula in the problem

A cylinder

A cylinder is an object with straight sides and circular ends of the same size. The volume of a cylinder can be found in the same way you find the volume of a solid with a matching base and top. The surface area of a cylinder can be easily found when you realize that you have to find the area of the circular base and top and add that to the area of the sides. If you slice the side of the cylinder in a straight line from top to bottom and open it up, you will see that it makes a rectangle. The base of the rectangle is the circumference of the circular base, and the height of the rectangle is the height of the cylinder

Volume of a cylinder

V=Ah

A = the area of the base of the cylinder

h = the height of the cylinder

Surface Area of a Cylinder

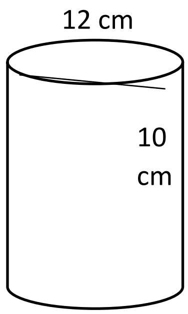
$SA=2(\pi r^2)+2\pi rh$

r = the radius of the circular base of the cylinder

h = the height of the cylinder

π = the number that is approximated by 3.141593

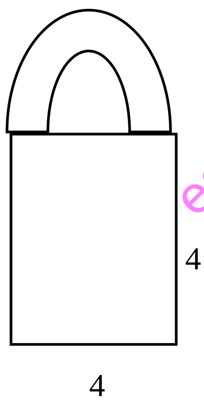
Find the area of the cylinder



$$SA = 2(\pi r2) + 2\pi rh$$
$$SA = 2(\pi.62) + 2\pi(6)(10) = 603.18579$$

8.3 Obtain Area of irregular figures

To find the area of irregular shapes, the first thing to do is to divide the irregular shape into regular shapes that you can recognize such as triangles, rectangles, circles, squares and so forth. Then, find the area of these individual shapes and add them up.



The figure above has two regular shapes. It has a square and half a circle. Find the area for each of those two shapes and add the results

Square

Area of square = s^2

Area of square = 4^2

Area of square = 16

Circle

Area of circle = pi × r^2

Notice that the radius of the circle is 4/2 = 2

Area of circle = $3.14 \times 2^{2=}3.14 \times 4$

Area of circle = 12.56

Since you only have half a circle, you have to multiply the result by 1/2
1/2 × 12.56 = 6.28
Area of this shape = 16 + 6.28 = 22.28

Example
Circle
To get the area of the half circle, we need to know the diameter. Notice that the diameter is the hypotenuse of a right triangle, so use the Pythagorean Theorem to find the length of the diameter

$c^2 = a^2 + b^2$
$c^2 = 122 + 162$
$c^2 = 144 + 256$
$c^2 = 400$
$c = \sqrt{400}$
$c = 20$

Therefore, the diameter is 20. Since the diameter is 20, the radius is 10

Area of circle = pi × r^2
Area of circle = 3.14 × 10^2
Area of circle = 3.14 × 100
Area of circle = 314

Since you only have half a circle, you have to multiply the result by 1/2

$1/2 \times 314 = 157$

Area of this shape = $384 + 96 + 157 = 637$

8.4 Obtain Areas and volumes using Pappus theorem
Pappus' centroid theorems are results from geometry about the surface area and volume of solids of revolution. These quantities can be computed using the distance traveled by the centroids of the curve and region being revolved.

Theorem
Let CC be a curve in the plane. The area of the surface obtained when CC is revolved around an external axis is equal to the product of the arc length of CC and the distance traveled by the centroid of CC.

Let RR be a region in the plane. The volume of the solid obtained when RR is revolved around an external axis is equal to the product of the area of RR and the distance traveled by the centroid of RR.
Consider the cylinder obtained by revolving a rectangle with horizontal side r and vertical side h around one of its vertical sides (say its left side). The surface area of the cylinder, not including the top and bottom, can be computed from Pappus' theorem since the surface is obtained by revolving its right side around its left side. The arc length of its right side is h and the distance traveled by its centroid is simply 2\pi r, 2πr, so its area is 2 \pi r h, 2πrh.

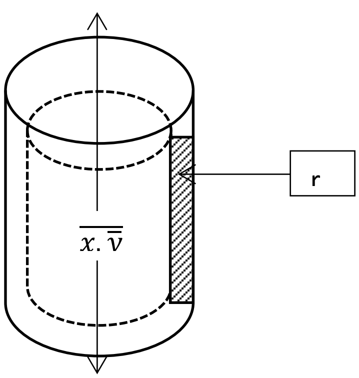
The volume of the cylinder is the area rh of the rectangle multiplied by the distance traveled by its centroid. The centroid of the rectangle is its center, which is a distance of r\2 from the axis of revolution. So it travels a distance of 2\pi\big (\frac r2\big) = \pi r2π (2r)=πr as it revolves. The volume of the cylinder is (rh) (\pi r) = \pi r^2 h= (rh)(πr)=πr^2h.

Example of volume of revolution

Theorem of Pappus

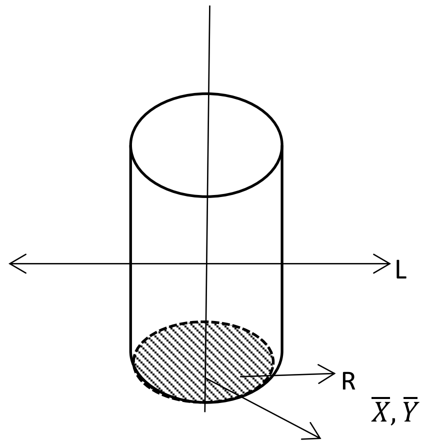
Let R be a region in a plane and let L be a line in the same plane such that L does not intersect the interior of R. If Ω is the distance between the centroid of R and the line, the volume of the solid of revolution R about the line is;

$V = 2\pi A$



X: $(\mathbf{MY} =)/\mathbf{M}$

Using Pappus' theorem to find the volume of the solid of revolution, the turns formed by revolution the circle.



$X^2 + (y+5)^2 = 9$

About the x-axis.

$V = 2\pi\ rA$

V=2πrA

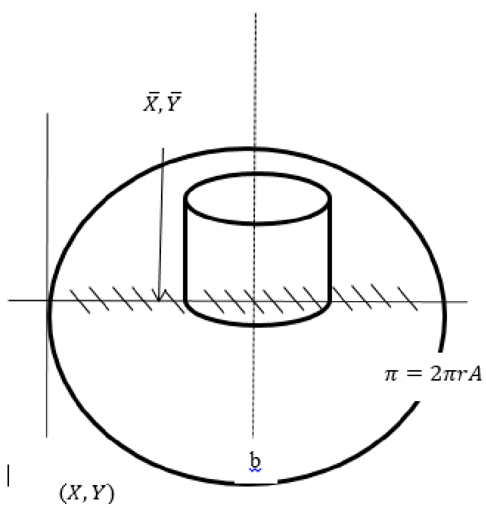
2(π)5(π(3)

90π

Example 2

Use the theorem of Pappus to find the volume of the solid of revolution formed by revolving the region bounded by the graph of:

Y=$\sqrt{X}$, Y = 0 andx = 4 about the line x = 6



$$\frac{\delta\int_a^b x[f(x)-g(x)]dx}{\delta\int_a^b x[f(x)-g(x)]dx} = \frac{my}{m} = \bar{x}$$

$$= \frac{\text{Moment about y-axis}}{\text{Mass of Lamina}}$$

Area of shaded region

A=$\int_{\theta}^{b}(x^{\frac{1}{3}} * dx$

$\left[\left[\frac{2}{3}\right] x^{-3}\right]$ -4
0

$\frac{2}{3}[4]^{\frac{3}{4}}$

$\frac{2}{3}[9]^{\frac{3}{4}}$

$\frac{2}{3}[8]^{\frac{3}{4}}$

$\frac{2}{3}(8) = \frac{16}{3}$

$= \ n = \frac{16}{3}, my = \frac{64}{5}$

$= \int_{\theta}^{4} x(x^{\frac{1}{3}})dx$

my=$\int_{\theta}^{4} x(x^{\frac{1}{2}})dx$

$= \int_{\theta}^{4} \left(x^{\frac{3}{2}}\right) dx$

$\left[\frac{2}{5} * \frac{5}{2}\right]$

=$\frac{2}{5}(4)^{\frac{5}{2}}$

=$\frac{2}{5}4^{\frac{5}{2}} - \frac{2}{5}(0)^{\frac{5}{2}}$

=$\frac{2}{5}(2)^{5}$

=$\frac{64}{5}$

$v = 2\pi rA$=2π($\frac{18}{5}$)($\frac{16}{3}$)

$= \frac{567}{15}\pi$

Conclusion

The learning outcome covered or equipped the learner with knowledge, skills and attitude to obtain perimeter and areas of figures, volume and surface area of solids, area of irregular figures and areas and volumes using Pappus theorem.

Further Reading

1. Hoyland, A., Rausand, and M. (1994), System Reliability Theory: Models and Statistical Methods, John Wiley (New York).

2.3.9.3 Self-Assessment

Written Assessment

1. An equilateral triangle of side length rr in the first quadrant, one of whose sides lies on the xx-axis, is revolved around the line y= -r. y=−r. The volume of the resulting solid is c\pi r^3cπr3 for some real number c.c. What is c.c?
2. Consider the single rectangle in R^2 that passes through the points A = (1,2), B = (2,1), C = (4,3), D = (3,4) A rotating around x-axis in R^3. The volume of the surface of revolution obtained can be written as A3 Aπ unit3. Submit A
3. Revolving a right triangle with legs of length r and h around the leg of length h produces a cone. The surface of the cone (not including the circular base) is obtained by revolving the hypotenuse around that leg. The centroid of the hypotenuse is just the midpoint, located halfway up the side of the cone, which travels a distance $2\pi/2$ as it rotates. So, the surface area is $2\pi r\sqrt{R2 + H2}$
4. Consider the cylinder obtained by revolving a rectangle with horizontal side r and vertical side h around one of its vertical sides (say its left side). The surface area of the cylinder, not including the top and bottom, can be computed from Pappus's theorem since the surface is obtained by revolving its right side around its left side. The arc length of its right side is h and the distance travelled by its centroid is simply2π, 2πr, so its area is 2πr h. 2πrh.

2.3.9.4 Tools, Equipment, Supplies and Materials

- Scientific Calculators
- Rulers, pencils, erasers
- Charts with presentations of data
- Graph books
- Computers with internet connection

2.3.9.5 References

Kaplan, W. (2000), Advanced Calculus, 3rd ed., Addison-Wesley (Cambridge, MA).
Kreyszig, E. (2004), Advanced Engineering Mathematics, 8th ed., John Wiley (New York).

2.3.10 Learning Outcome No 9: Apply Power Series

2.3.10.1 Learning Activities

Learning Outcome No 9: Apply Power Series	
Learning Activities	**Special Instructions**
9.1 Obtain power series using Taylor's Theorem 9.2 Obtain power series using McLaurin's theorem	

2.3.10.2 Information Sheet No2/LO9: Apply power series

Introduction

This learning outcome covers; derivation of power series using Taylor's Theorem and derivation of power series using McLaurin's theorem.

Content/Procedures/Methods/Illustrations

9.1 Obtain power series using Taylor's Theorem

The power series of McLaurin's theorem is different functions can be carried out using two theorems.

(i) Taylor's Theorem
(ii) Maclaurin's theorem

Taylor's series states that;

$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ ...$

Examples

Express $\text{Sin}(x+h)$ as a series of powers of h and hence evaluates $\text{Sin } 44^0$ correct to four decimal places.

Solution

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots$$

$f(x) = \text{Sinx}$

$f'(x) = \cos x$

$f''(x) = -\text{sinx}$

$f'''(x) = \text{-cosx}$

$f^{iv}(x) = \sin x$

$\therefore$ $\text{Sin}(x + h) = \sin x + h\cos x - \frac{h^2}{2}\sin x - \frac{h^3}{6} \cos x$ ....

$\text{Sin}44^0 = \sin(45^0 - 1^0)$
$=\text{Sin}(^{\pi}/_{4}) - 0.01745$
$=\sin {}^{\pi}/_{4} + 0.01745 \cos\frac{\pi}{4} - \frac{0.01745^2}{2}\sin {}^{\pi}/_{4} - \frac{0.0174s^3}{6} \cos {}^{\pi}/_{4}$

But $\sin 45 = \cos 45 = 0.707$
$= 0.707\ (1 - 0.01745 - 0.0001523 + 0.0000009)$
$= 0.707(0.982395)$
$= 0.69466$
$0.6947(4dp)$
For the use McLaurin's theorem refer to Engineering Mathematics by Strand.

9.2 Obtain power series using McLaurin's theorem

McLaurin's series is a special type of Taylor series where b is centred around where the variable x or y is zero. That is; from the Taylor series.
$f(x) = f(b) + (f^{\wedge\prime}(b)(x-b))/1! + (f''(b)(x-b)^2/2! + \cdots$ We obtain the following McLaurin's series:

f(x)= f(0)+ $f^1(0)(x) + \frac{f''(0)(x)^2}{2!} + \cdots$

How to find the McLaurin's series.

i. Get the derivatives of the given function until you notice a particular pattern

For example

$p(x) = \sin x$
$p'(x) = \cos x$
$p''(x) = -\sin x$
$p'''(x) = -\cos x$

$p'^{v}(x) = \sin x$

Note that $p'(x) = p^{v}(x)$
$p''(x) = p^{vi}(x)$
$p'''(x) = p^{vii}(x)$ and so on

ii. Replace the variable in the first step with 0 as the input to get the McLaurin's series values.

That is;

$p(x) = p(0) + p'(0)x^1 + p''(0)\frac{x^2}{2!} + p'''(0)\frac{x^3}{3!} + \cdots$

$sin\ x$ = 0+ x+0 $- \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + \cdots$

iii. Write what your get in step (ii) in sigma notation.
This is will enable one to obtain the final power series.
That is;

$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} -$

$=\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}(-1)^n$

Example.

Obtain the McLaurin's series for $e^{5y.}$

$f(y)$=5$e^{5y.}$

f'(y)=5$e^{5y.}$

f''(y)=25$e^{5y.}$=$5^2e^{5y.}$

f'''(y)=125$e^{5y.}$=$5^3e^{5y.}$

$f(y)$=$e^{5(0).}$=1

f' (y)=5$e^{5(0)} = 5$

f''(y)=25$e^{5(0)} = 5^2$

f'''(y)=125$e^{5(0)} = 125 = 5^3$

$e^{5y} = 1 + \frac{5(y-0)}{1!} + \frac{5^2(y-0)^2}{2!} + \frac{5^3(y-0)^3}{3!}$

$= \sum_{n=0}^{\infty} 5^n \frac{y^n}{n!}$

Some common examples of the McLaurin's series include:

$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$

$=\sum_{n=0}^{\infty} x^n$

$f(x) = e^x = 1 + x + \frac{1}{2^{x^2}} + \frac{1}{6^{x^3}} + \frac{1}{24^{x^4}} + \cdots$

$=\sum_{n=0}^{\infty} \frac{1}{n!} x^n$

Conditions for McLaurin's series.

- The derivative of the given function, say q(x) must exist for all n i.e $q^n(x)exists$ for all $n = 1,2 \ldots\ldots n$
- $f^n(0)$must be defined for all n $= 1,2 \ldots\ldots,$ n
- The series obtained should converge to the given function say q(x).

Conclusion

The learning outcome covered or equipped the learner with knowledge, skills and attitude to derive power series using Taylor's Theorem and power series using McLaurin's theorem.

Further Reading

1. Greenberg, M.D. (1998), Advanced Engineering Mathematics, 2nd ed., Prentice Hall (Upper Saddle River, N.J).
2. Hildebrand, F.B. (1974), Introduction to Numerical Analysis, 2nd ed., McGraw-Hill (New York).
3. Hildebrand, F.B. (1976), Advanced Calculus for Applications, 2nd ed., Prentice-Hall (Englewood Cliffs, NJ).

2.3.10.3 Self-Assessment

Written Assessment

1. Use McLaurin's theorem to expand $\ln(3x+1)$. Hence use the expansion to evaluate $\int_0^1 \frac{\ln(3x+1)}{x^2}dx$ to four decimal places.
2. Use Taylor's series to expand $\cos\left(\frac{\pi}{3}+h\right)$ in terms of h as far as h^3. Hence evaluate $\cos 68^0$ correct to four decimal places.

2.3.10.4 Tools, Equipment, Supplies and Materials

- Scientific Calculators
- Rulers, pencils, erasers
- Charts with presentations of data
- Graph books
- Dice
- Computers with internet connection

2.3.10.5 References

Hoyland, A., Rausand, and M. (2005), System Reliability Theory: Models and Statistical Methods, John Wiley (New York).

Kaplan, W. (2003), Advanced Calculus, 3rd ed., Addison-Wesley (Cambridge, MA).

2.3.11 Learning Outcome No 10: Apply Statistics

2.3.11.1 Learning Activities

Learning Outcome No 10: Apply Statistics	
Learning Activities	**Special Instructions**
1.1 Perform identification, collection and organization of data. 1.2 Perform interpretation, analysis and presentation of data in appropriate format. 1.3 Obtain mean, median, mode and standard deviation from given data 1.4 Perform calculations based on laws of probability 1.5 Perform calculation involving probability distributions, mathematical expectation sampling distributions 1.6 Apply sampling distribution methods in data analysis 1.7 Perform calculations involving use of standard normal table, sampling distribution, t-distribution and estimation 1.8 Determine confidence intervals 1.9 Perform testing hypothesis using large samples and small samples 1.10 Do calculations involving correlation and regression 1.11 Do calculations involving rank correlation coefficient and equations of regression line	• Illustrations • Oral questioning • Written tests • Assignments • Supervised exercise

2.3.11.2 Information Sheet No2/LO10: Apply Statistics

Introduction

This learning outcome covers classification of data, grouped data, ungrouped data, data collection, tabulation of data, class intervals, class boundaries, frequency tables, diagrammatic and graphical presentation of data e.g. histograms, frequency polygons, bar charts and pie charts. It also covers cumulative frequency curves, measures of central tendency mean, mode and median, measures of dispersion, variance and standard deviation, definition of probability, laws of probability, expectation variance and S.D, types of distributions, mean, variance and SD of probability distributions, application of probability distributions, standard normal tables and sampling distributions and rank correlation coefficient.

Definition of key terms

Mode: This is the number which is the most repeated in a series.

Standard deviation: This is the amount of variation of a set of numbers. It is the square root variance.

$$\sigma = \sqrt{\frac{\sum(x_i - \mu)^2}{N}}$$

$$\partial - \ Standard\ Deviation$$
$$\mu - \ Mean$$
$$x_1 - Each\ Value$$
$$N - Number\ of\ values$$

Variance: This is the mean of the aquared differences of the number from the mean.

$$\sigma^2 = \frac{\Sigma(\,x - \mu)^2}{N}$$

Where N – size of the number

$$\mu - Mean$$
$$x_i - Each\ value\ (Number)$$

Content/Procedures/Methods/Illustrations

1.1. Identification, Collection and Organization of data is performed

Data identification is described as the records that links the value to give more sensible information or database such as age. Data collection on the other hand is the process of acquiring information of the targeted variables in the system such as the ages of children in class while data organisation refers to way of classifying where there are two main type which grouped data and individual data.

1.2. Interpretation, analysis and presentation of Data in appropriate format is performed

Data interpretation

This means analysing of data to infer information which enables proper answering of the relevant question.

Data analysis

This means the process of cleaning and transforming the inspected data to model data with an objective to achieve.

Data presentation

This means the arrangement of data into graphs, tables and charts. The data can be classified as either grouped or ungrouped data.

Ungrouped data: It is the data (first data) gathered from a study or experiment.

Digrammatic and graphical presentation of Data

Histagram: IIt is an adequate representation of data and how it is distributed.
It is represented with a table which helps in showing the probability.

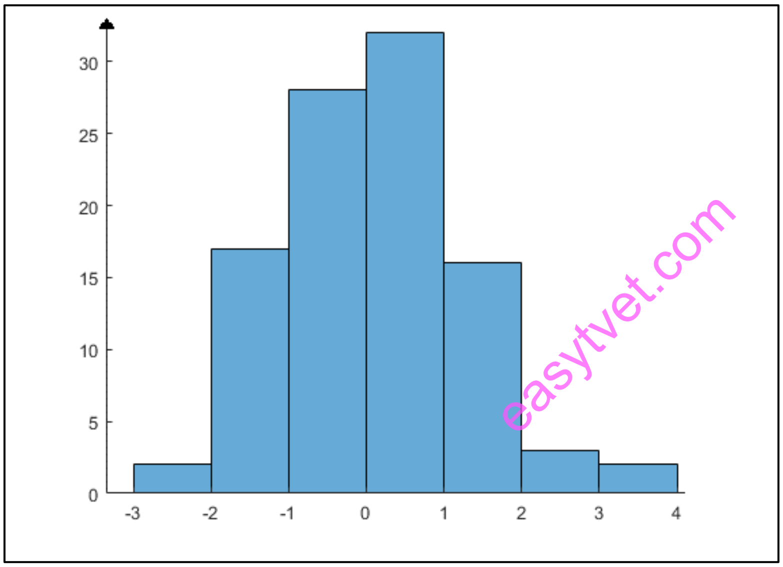


Figure 3: Histogram

Frequency polygon- This is a graph constructed by straight line passing on the midpoint of the class.

For example:

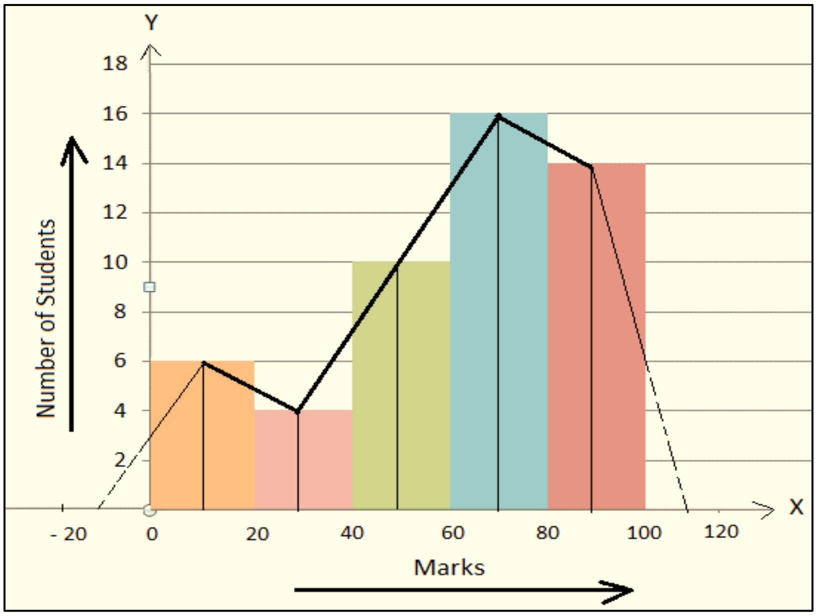


Figure 4: Frequency Polygon

Bar chart: It is a graph or chart that uses rectangles and their differences in heights or length to present data.

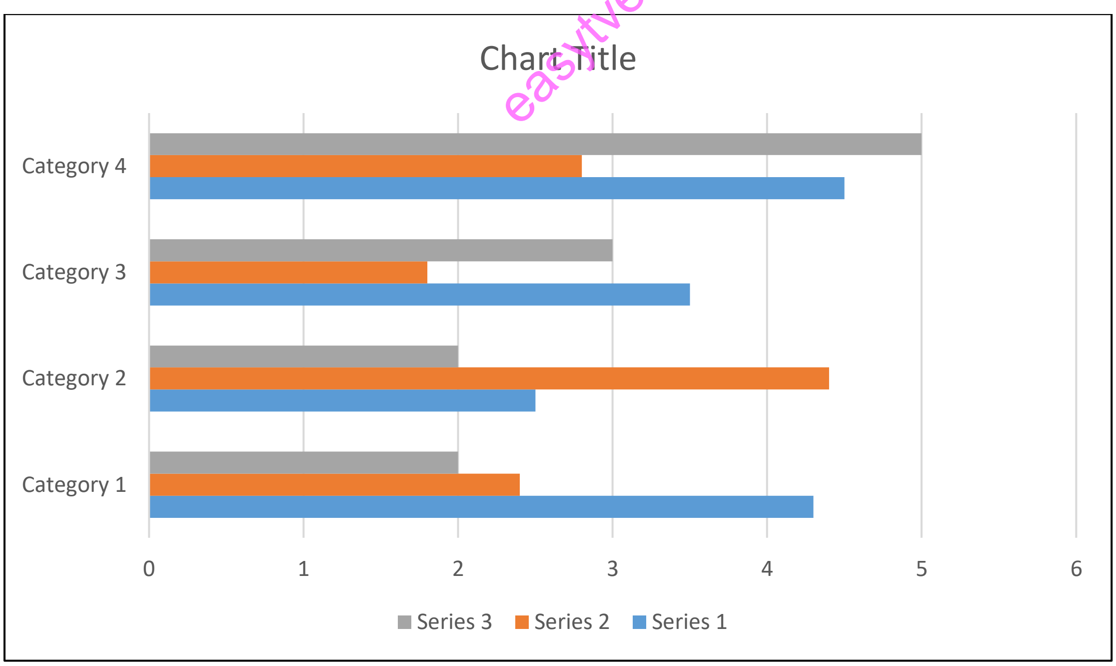


Figure 5: Bar Graph

Pie chart: This is a type of chart which represents data on a circle or pie where the pie is the total number.

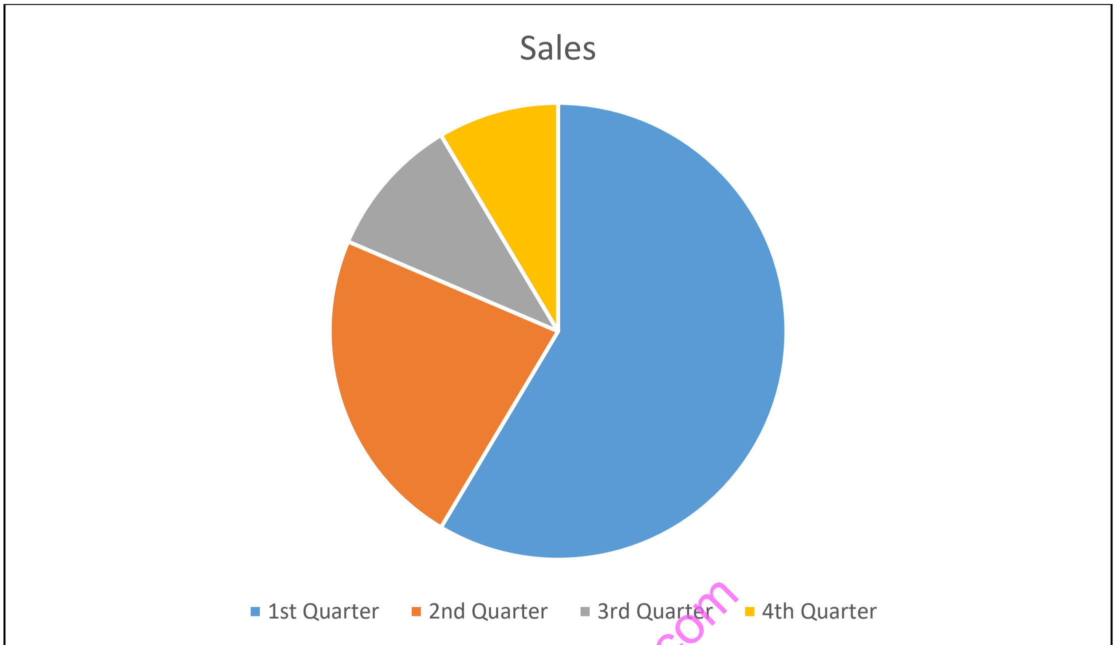


Figure 6: Pie Chart

Cumulative frequency curves: This is a curve which shows the cumulative frequency distribution of data mainly grouped data.

Data tabulation

Class intervals; this is the size of the individual group of data.

Class boundaries; these are the upper limit or the lower limit of grouped data.

Frequency table; it is a representation of frequency of various outcomes in a sample.

1.3. Median mode and Standard deviation are obtained from given data

Mean: It is also known as average therefore it is addition of the number divided by the number of the numbers.

$$Mean = \frac{\sum of\ numbers}{no\ of\ numbers}$$

Median: This is the number in the middle after being arranged from the lowest to the highest number; if they are two, find the mean.

Mode: This is the number which is the most repeated in a series.

Variance: This is the mean of the squared differences of the number from the mean.

$$Variance = \sum(x_{1-}\mu)^2/n$$

Standard deviation: This is the amount of variation of a set of numbers known as square-root variance.

1.4. Calculations are performed based on Laws of probability

Definition of probability; is a description of how likely an event or occasion to occur or how likely to happen or true.

Laws of probability

The Law of large numbers is the principle that the more trials you have in an experiment, the higher you get to the accurate value in probability. E.g a set of cards, probability to get black cards in two take is $1/52x2 = 1/26$

$$P(face)$$

Addition rule

Based on the next turn $P(AuB) = P(A) + P(B) - P(AnB)$

Multiplication Rule

It deals with the case in and of probabilities. It means the probability of two independent events.

$P(A\ and\ B) = P(A).P(B/A) = P(AnB)$

Example 2

A bag contains 3 pink candies and 7 green candles. 2 candies are taken out from the bag with replacement. Find the probability that both candies are pink.

Solution

Let A= event that 1st candy is pink and B= event that second candy is pink

$$p(A) = 3/10$$

$$A\ and\ B\ are\ independent$$

$$P(B/A) = P(B) = 3/10$$

Therefore, multiplication law we got

$P(AnB) = P(A)x\ P(B/A) \quad 3/10x3/10 = 9/100 = 0.09$

1.5. Calculation involving probability distributions, mathematical expectation sampling distributions are performed

Probability distribution is all the lekely and possible values that a random variable can take within the range. Also known as the 'bell curve'

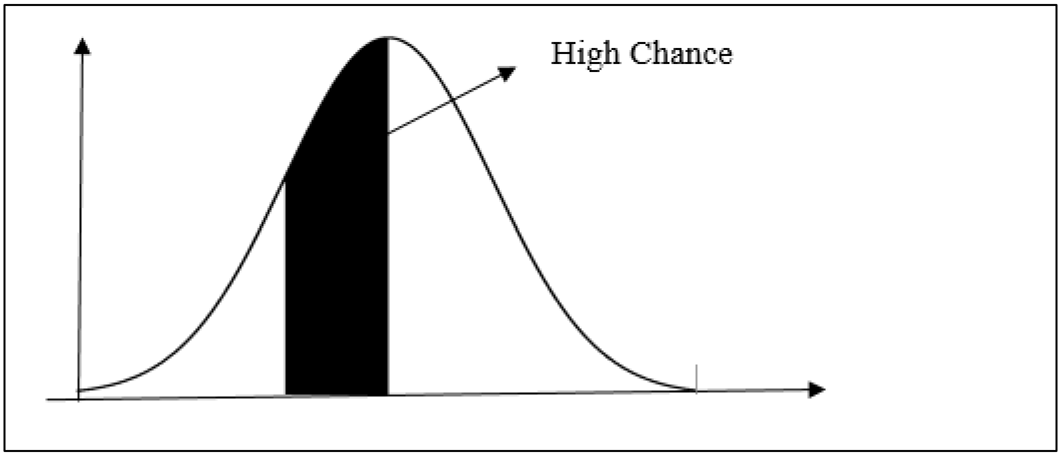


Figure 7: Probability Distribution

There are 4 common statistics in sampling distribution; the sample sum, the sample mean, the sample variance.

Sample sum is the sum of a random variables from the data

$$X_1 + \;\; X_2 \ldots, \;\; X_n$$

Sample mean is the average of the random variables:

$$M = (X_{1+}X_{2\ldots,X_n})/n$$

$$S^2 = ([(X_1 - M)^2 + (X_2 - M)^2 \ldots, ((X_n - M)^2])/(n-1)$$

Note square root estimates the standard deviation of a population.
Therefore the bell curve is from the standard deviation to the mean.

Example

Draw and show the bell curve if the mean is 30 and the standard deviation is 26.
The curve for the example is shown in the next page.

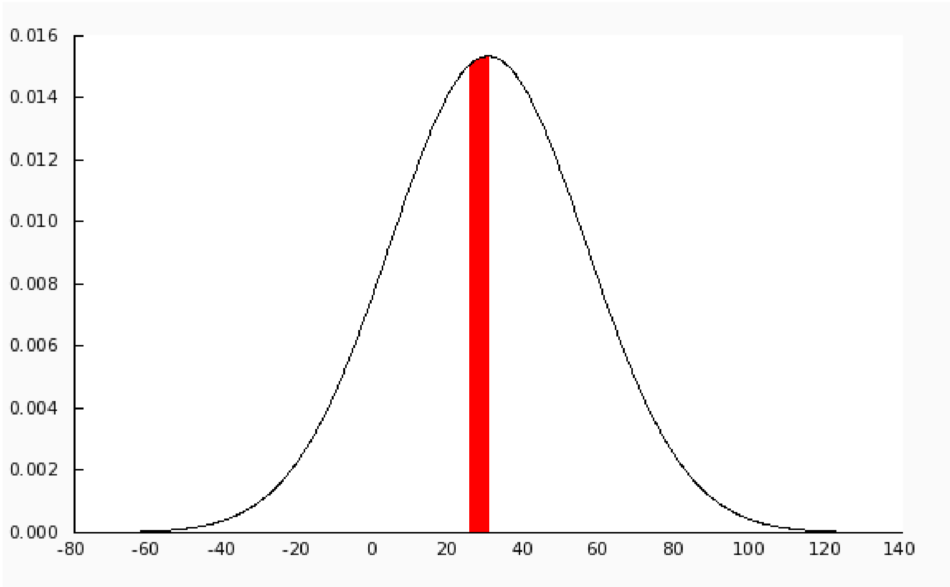


Figure 8: Bell curve

1.6 Sampling Distribution methods are applied in Data Analysis

There are three main types;

- **Normal distribution;** commonly used in investing, finance, science and engineering. It fully based on its mean and standard deviation.
- **Binomial distribution**: It is discrete, as opposed to continuous, since 1 or 0 / yes or no is a valid response.
- Chi-squared distribution
- Poisson distribution

NOTE: The first and second types are the most common used during data analysis in Kenya.

1.7 Calculations involving use of standard normal table, sampling Distribution, T-distribution and Estimation are done

Standard normal table (known Z table)

$$Z = \frac{x - \mu}{\alpha}$$

$$X = raw\ score$$

$$\mu = mean$$

$$\alpha\ = standard\ deviation$$

A Z found on the both table should be used on the table provided to known the percentile and therefore compare.

Note if it is a negative then a table for negative values is used.

Sampling distribution

It is also known as probability distribution and the standard deviation of this topic is known as standard error.

Samplig distribution mean is equal to the mean of the population.

$\mu_x = \mu$

Therefore standard error is:

$$\sigma = \sqrt{\frac{\sum(x_i - \mu)^2}{n-1}}$$

$$SE = \frac{\sigma}{\sqrt{n}}$$

SE Standard error

∂ =Standard deviation

μ=Size of population

n= size of sample

If $fpc = 1$ from factor$\sqrt{N-n)/N-1)}$, therefore standard arror formula can be approximated by

$$\sqrt{(\mathrm{x} = \alpha)}/\sqrt{n}$$

Distribution

It is a type of distribution similar with the normal distribution curve but with a bit shorter and fatter tail. Therefore distribution is used because small size is small.

$$t = \frac{x - \mu}{(\frac{s}{\sqrt{n}})}$$

Where,

$\bar{x}$ is the sample mean

μ is the population mean

s is the standard deviation

n is the size of the sample given

Estimation

This is the process of identifying a value by approximating due to a certain purpose. It can be done by rounding off to the nearest whole number.

1.8 Confidence intervals are determined

It is a type of estimate computed from the statistics of the observed data It is a range of value where a true value lies on.

Calculation

1) Find mean $\bar{x}$ and standard deviation s
2) Find 2 m table to find the percentile

   $Z=\overline{x - \mu/\alpha}$

3) Using z un the formula for the confidence interval.

   $\overline{x} \pm z\left(\frac{s}{n}\right)$

1.9 Testing hypothesis using large samples and small samples are performed

It is an act whereby an analyst test an assumption regarding a population parameter.

They are two types

a) Null hypothesis is equal to zero
b) Alternative hypothesis is not equal to zero

Steps of hypothesis testing

a) Analyse the state the 2 hypothesis so that only can be right.
b) Formulating an analysis plan, outline how the data will be evaluated.
c) Carry out the plan and physically Analyse the data.
d) Analyse the result either reject or accept if it is null hypothesis.

1.10 Calculations involving Correlation and regression are done

They are related because both deal with relationships among variable.

Where correlation is a measure of linear association between 2 variables while regression involves identifying the relationship between a dependent variable and one or more independent variable.

Calculation of correlation

$$r = \sum(x - \overline{x)(y - y/\sqrt{\{\sum(x - \bar{x})^2}})x^2(y - y)^2)$$

1.11 Calculations involving rank correlation coefficient and equations of regression line are done

Rank correlation coefficient: It is a tool to discover the strength of link between sets of data.

Method

a) Create a table
b) Rank the two data sets
c) Tied scores are given the mean rank
d) Square the differences

Linear regression

Formula

$Y = a + bx$

$Y = explanatory\ value$

$X = dependent\ value$

$b = slope\ line$

$a = intercept$

Conclusion

This learning outcome has covered classification of data, grouped data, ungrouped data, data collection, tabulation of data, class intervals, class boundaries, frequency tables, diagrammatic and graphical presentation of data e.g. histograms, frequency polygons, bar charts and pie charts. It also covered cumulative frequency curves, measures of central tendency mean, mode and median, measures of dispersion, variance and standard deviation, definition of probability, laws of probability, expectation variance and S.D, types of distributions, mean, variance and SD of probability distributions, application of probability distributions, standard normal tables and sampling distributions and rank correlation coefficient.

Further Reading

Read more on:

1. Correlation from the book Cox. D, Principles of applied statistics, (2011).
2. Confidence interval from the internet.

2.3.2.3 Self-Assessment

Written Assessment

1. Which of the following is NOT a way of presenting data?
    a) Figures
    b) Chart
    c) Graph
    d) Table
2. Suppose the covariance between Y and X is 12, the variance of Y is 25, and the variance of X is 36. The correlation coefficient, r, between Y and X is closest to:
    a) r = 0.000
    b) r = 0.013
    c) r = 0.160
    d) r = 0.400
3. The following represents age distribution of students in an elementary class. Find the mode of the values: 7, 9, 10, 13, 11, 7, 9, 19, 12, 11, 9, 7, 9, 10, 11.
    a) 7
    b) 9
    c) 10
    d) 11
4. Find the mean of 8, 5, 7, 10, 15, and 21
    a) 15
    b) 6
    c) 11
    d) 4
5. The standard deviation of a sample of 100 observations equals 64. The variance of the sample equals
    a) 8
    b) 10
    c) 6400
    d) 4096
6. Which of the following is not a measure of dispersion
    a) Range
    b) The 5th percentile
    c) The standard deviation
    d) The interquartile range

7. In a certain game, players toss a coin and roll a dice. A player wins if the coin comes up heads, or the dice with a number greater than 4. In 20 games, how many times will a player win?

    a) 13
    b) 8
    c) 11
    d) 15

8. Justify the use of binomial distribution in Modern mathematics?
9. Classify the laws of probability?
10. A box contains 30 red, green and blue balls. The probability of drawing a red ball is twice the other colours due to its size. The number of green balls are 3 more than twice the number of blue balls, and blue are 5 less than the twice the red. What is the probability that 1st two balls drawn from the box randomly will be red?
11. There are 3 blue, 1 white and 4 red identical balls inside a bag. If it is aimed to take two balls out of the bag consecutively, what is the probability to have 1 blue and 1 white ball?
12. Kamau has two children and we know that she has a daughter. What is the probability that the other child is a girl as well? Describe the term rank correlation coefficient?

Oral Assessment

1. The average age of 6 persons living in a house is 23.5 years. Three of them are majors and their average age is 42 years. The difference in ages of the three minor children is same. What is the mean of the ages of minor children?
2. The arithmetic mean of a set of 10 numbers is 20. If each number is first multiplied by 2 and then increased by 5, then what is the mean of new numbers?

Practical Assessment

In a group analyses the data and form a table of the type of cars passed in the nearest road according to type.

2.3.11.4 Tools, Equipment, Supplies and Materials

- Scientific Calculators
- Rulers, pencils, erasers
- Charts with presentations of data
- Graph books
- Computers with an internet connection

2.3.11.5 References

Bland, J. M., & Altman, D. G. (2010). Statistical methods for assessing agreement between two methods of clinical measurement. International Journal of Nursing Studies, 47(8), 931-936.

McCuen, R. H. (2002). Approach to confidence interval estimation for curve numbers. Journal of Hydrologic Engineering, 7(1), 43-48.

Torabi, H., & Behboodian, J. (2007). Likelihood ratio tests for fuzzy hypotheses testing. Statistical Papers, 48(3), 509.

2.3.12 Learning Outcome No 11: Apply numerical methods
2.3.12.1 Learning Activities

Learning Outcome No 11: Apply numerical methods	
Learning Activities	**Special Instructions**
1.1. Obtain roots of polynomials using iterative numerical methods (Newton Raphson and Gregory Newton)	

2.3.12.2 Information Sheet No2/LO11: Apply numerical methods

Introduction

This learning outcome covers Application of numerical methods, roots of polynomials, and performing interpolation and extrapolation using numerical methods.

Definition of key terms

Numerical methods: Are algorithms used for computing numeric data.

Content/Procedures/Methods/Illustrations

1.1 Obtain roots of polynomials using iterative numerical methods (Newton Raphson and Gregory Newton)

Numerical method is a complete and definite set of procedures for the solution of a problem, together with computable error estimates. The study and implementation of such methods is the province of numerical analysis.

Types of numerical methods

a) Bisection method
b) Newton Raphson method (Newton's Iteration method)
c) Iteration method
d) Newton's forward interpolation formula
e) Newton's backward interpolation formula
f) Gauss Seided method
g) Curve fitting

Applications

- Used in computer science for root algorithm
- Used to determine profit and loss in the company
- Solving practical technical problems using scientific and mathematical tools

- Used for multidimensional root finding
- Network simulation
- Train and traffic signal
- Weather prediction
- Build up an algorithm

Worked example

Construct a difference table to find polynomial of the data $(1,1), (2,8), (3,27), (4,64), (6,216), (7,343), 8,512)$.
Considering appropriate method find r, where (9,r) given.

Solution

We may construct anyone of forward backward and central difference tables. Since we also have to; Find r for x=9, which is nearer at the end of the set of given tabular values, so we will construct the backward difference table.

Table 1. The backward difference table of the data

X	Y	$\overline{\nabla}$	∇^{-2}	∇^{3}	∇^{4}
1	1	7	12	6	0
2	8	19	18	6	0
3	27	37	24	6	0
4	64	61	30	6	0
5	125	91	36	6	
6	216	127	42		
7	343	167			
8	512				

This the required difference table:
Here:
$X_n = 8, h = 1, Y_n = 512$
$\nabla^{-2}Yn = 42, \nabla^{3}Yn = 6, \nabla^{4}Yn = 0$

$$\text{Therefore ; } P = \frac{x - xn}{n} = \frac{x-8}{1} = (x-8)$$

By newtons backward formular;

$$Y(X) = Yn + p\nabla_{yn} + p\frac{(p+1)\nabla_{yn}}{2} + \frac{p(p+1)(p+2)\nabla_{yn}^{1}}{3!}\text{-------}+\frac{p(p+1)(p+n-1)\nabla_{yn}^{n}}{n!}$$

$$= 512 = 169\frac{(x-8)+(x-8)(x-8+1)*42}{2!}$$

Conclusion

The learning outcome covered apply numerical methods, roots of polynomials, and perform interpolation and extrapolation using numerical methods.

Further Reading

2.3.12.4 Tools, Equipment, Supplies and Materials

- Scientific Calculators
- Rulers, pencils, erasers
- Charts with presentations of data
- Graph books
- Dice
- Computers with internet connection

2.3.12.5 References

Greenberg, M.D. (1998), Advanced Engineering Mathematics, 2nd ed., Prentice Hall (Upper Saddle River, N.J).

Hildebrand, F.B. (2003), Introduction to Numerical Analysis, 2nd ed., McGraw-Hill (New York).

Hildebrand, F.B. (2002), Advanced Calculus for Applications, 2nd ed., Prentice-Hall (Englewood Cliffs, NJ).

2.3.13 Learning Outcome No12: Apply Vector Theory

2.3.13.1 Learning Activities

Learning Outcome No 12: Apply Vector Theory	
Learning Activities	**Special Instructions**
12.1 Obtain vectors and scalar quantities in two and three dimensions 12.2 Perform Operations (addition and subtraction) on vectors 12.3 Obtain position of vectors 12.4 Work out resolution of vectors	Encourage students to practice

2.3.13.2 Information Sheet No2/LO12: Apply Vector Theory

Introduction

This learning outcome covers vectors and scalar quantities in two and three dimensions, operations on vectors, and position and resolution of vectors.

Definition of key terms

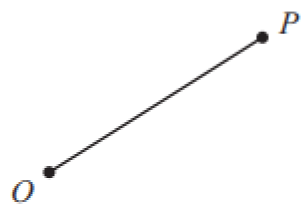
Vectors: A quantity having direction as well as magnitude, especially as determining the position of one point in space relative to another. A vector has both magnitude and direction, and both these properties must be given in order to specify it. A quantity with magnitude but no direction is called a scalar.

Content/Procedures/Methods/Illustrations

12.1 Apply Vector theory

Physical quantities can be divided into two main groups, scalar quantities and vector quantities. A Scalar quantity is one that is defined completely by a single number with appropriate units e.g. Lengths, area, volume, mass, time etc. A Vector quantity is defined completely when we know not only its magnitude but also the direction in which it operates, e.g. force, velocity, acceleration, etc.

Vector quantities are extremely useful in physics. The important characteristic of a vector quantity is that it has both a magnitude (and size) and a direction. Both of these properties must be given in order to specify a vector completely. An example of a vector quantity is a displacement. This tell us how far away we are from a fixed point, and it also tells us our direction relative to that point.



Another example of a **vector quantity** is **velocity**. This is speed, in a particular direction. An example of velocity might be 60 mph due north. A quantity with magnitude alone, but no direction, is not a vector. It is called a scalar instead. One example of a **scala**r is **distance**. This tells us how far we are from a fixed point, but does not give us any information about the direction. Another example of a scalar quantity is the mass of an object.

12.2 Representing vector quantities

Represent a vector by a line segment

This diagram shows two vectors.

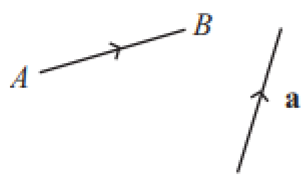
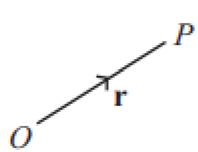


Figure 9. Vectors

The small arrow indicate that the first vector is pointing from A to B. A vector pointing from B to A would be going in the opposite direction.

Position vectors

Position vectors are referred to as fixed point, an origin. The position vector of a point P with respect to an origin O. In writing, might put OP for this vector. Alternatively, we could write it as **r**. These two expressions refer to the same vector.



Notation for vectors

What does it mean if, for two vectors, **a** = **b**? This means first that the length of **a** equals the length of **b**, so that the two vectors have the same magnitude. But it also means that **a** and **b** are in the same direction. How can we write this down more succinctly?

If two vectors are “in the same direction”, then they are parallel. We write this down as **a**//**b**.

For length, if we have a vector AB, we can write its length as AB without the bar. Alternatively, we can write it as |AB|. The two vertical lines give us the modulus, or size of, the vector. If we have a vector written as **a**, we can write its length as either |**a**| with two vertical lines, or as **a** in ordinary type (or without the bar). This is why it is very important to keep to the convention that has been adopted in order to distinguish between a vector and its length.

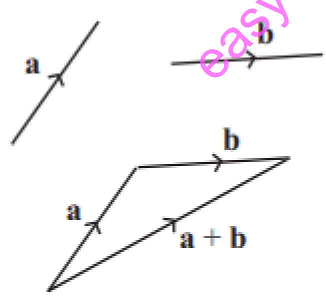
The length of a vector AB is written as **AB or |AB|,** and the length of a vector a is written as

a (in ordinary type, or without the bar) or as |**a**|

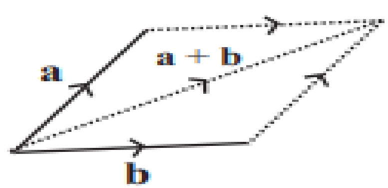
If two vectors **a** and **b** are parallel, we write **a//b**

12.3 Adding two vectors

In order to add two vectors, we think of them as displacements. We carry out the first displacement, and then the second. So, the second displacement must start where the first one finishes.



The sum of the vectors, a + b (or the resultant, as it is sometimes called) is what we get when we join up the triangle. This is called the triangle law for adding vectors. There is another way of adding two vectors. Instead of making the second vector start where the first one finishes, we make them both start at the same place, and complete a parallelogram. This is called the parallelogram law for adding vectors. It gives the same result as the triangle law, because one of the properties of a parallelogram is that opposite sides are equal and in the same direction, so that b is repeated at the top of the parallelogram.



Refer to Engineering mathematics by K. A Stroud to learn more on components of 0 Vector in terms of unit Vectors on page 368. Dot and cross product of vectors. The Scalar product of two vectors is denoted by $\bar{a}.\bar{\mathbf{b}}$ (sometimes called the 'dot product'.
The dot product of two vectors is defined as $\mathbf{a.b} = |\mathbf{a}||\mathbf{b}|\mathbf{cos\theta}$ where θ is the angle between **a** and **b**.

Refer to Technician mathematics 3 by J.O. Bird on page 297.

Examples
Solution
$\bar{\mathbf{a}} = \mathbf{2i + 3j + 5k}$ **And**$\bar{\mathbf{b}} = \mathbf{4i + j + 6k}$, $\bar{\mathbf{a}}.\bar{\mathbf{b}}$
$\bar{a}\ .\ \bar{b} = 2.4 + 3.1 + 5.6$
= 8 + 3+ 30
=41

A typical application of scalar products is that of determining the work done by a force when moving a body. The amount of work done is the product of the applied force and the distance moved in the direction of the applied force.
Example
Find the work done by a force F newtons acting at point A on a body, when A is displaced to point B, the coordinates of A and B being (3, 1, -2) and (4, -1, 0) metres respectively and when
$F = -i - 2j - k$ Newton's.

Solution
If a vector displacement from A to B is d, then the work done is F. d Newton Meters or joules. The position vector OA is $3i + j - 2k$ and OB is $4i - j$

$AB = d = OB - OA$
$= (4i - j) - (3i + j - 2k)$
$i - 2j + 2k.$
Work done $= F.d = (-1)/1) + (-2)(-2) + (-1)(2)$
$= -1 + 4 - 2$
=1 Nm or joule
For more worked examples refer to Technician mathematics 3 by J. O. Bird.

Cross Product

The vector or Cross product of two vectors $\bar{a}$ and $\bar{b}$ is C where the magnitude of C is $|\bar{a}||b|\text{Sin}\theta$ where θ is the angle between $\bar{a}$ and b.

For more information refer to Technician mathematics 3 by J.O Bird and Engineering mathematics by K. A Stroud.

Examples

$\bar{p} = 2i + 4j + 3k$ and $Q = i + 5j - 2k$ find $\overline{P} \times \overline{Q} = \begin{vmatrix} i & j & k \\ 2 & 4 & 3 \\ 1 & 5 & -2 \end{vmatrix}$

$= i\begin{vmatrix} 4 & 3 \\ 5 & -2 \end{vmatrix} - j\begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} + k\begin{vmatrix} 2 & 4 \\ 1 & 5 \end{vmatrix}$

$= -23i + 7j + 6k$

Typical applications of vector products are to moments and to angular velocity.

Refer to Technician mathematics. 3 by J.O Bird on page 308.

12.4 Vector field Theory

Refer to further Engineering mathematics by K.A Stroud to learn and also go through the worked examples and exercises on:

(i) Gradient

(ii) Divergence

(iii) Curl

Greens theorem: Learn how to perform vector calculations using Green's theorem by referring to further Engineer mathematics by KA. Stroud.

Stoke's Theorem: Refer to further Engineer Mathematics by K.A Stroud to learn how to perform vector calculations using Stroke's theorem.

Gauss's Theorem: Refer to the same book to learn how to determine line and surface integrals using Gauss's theorem.

Conclusion

This learning outcome covered vectors and scalar quantities in two and three dimensions, operations on vectors, and position and resolution of vectors.

Further Reading

1. Greenberg, M.D. (1998), Advanced Engineering Mathematics, 2nd ed., Prentice Hall (Upper Saddle River, N.J).

2.3.13.3 Self-Assessment

Written Assessment

1. If $\bar{a} = 2i - 3j + 4k$ and $\bar{b} = i + 2j + 5k$ determine

   $(\acute{2})$ $\bar{a} \cdot \bar{b}$

   $(2\acute{2})$ $\bar{a} \times \bar{b}$

2. Find the work done by a force F Newtons acting at a point A on a body, when A is displaced to point B ,the coordinates of A and B being (5, 2, -4) and (3,-1,1) meters respectively , and when $F = -2i - 3j - 2k$ Newton's.

2.3.13.4 Tools, Equipment, Supplies and Materials

- Scientific Calculators
- Rulers, pencils, erasers
- Charts with presentations of data
- Graph books
- Dice
- Computers with internet connection

2.3.13.5 References

Duffy, D. G. (2016). Advanced engineering mathematics with MATLAB. Chapman and Hall/CRC.

Jeffrey, A. (2001). Advanced engineering mathematics. Elsevier.

Zill, D., Wright, W. S., & Cullen, M. R. (2011). Advanced engineering mathematics. Jones & Bartlett Learning.

2.3.14 Learning Outcome No 13: Apply Matrix

2.3.14.1 Learning Activities

Learning Outcome No 11: Apply Matrix	
Learning Activities	**Special Instructions**
11.1 Obtain determinant and inverse of 3x3 matrix 11.2 Obtained Solutions of simultaneous equations in three unknowns are 11.3 Perform Calculation involving Eigen values and Eigen vectors	• Discussions • Projects • Demonstration by the trainer

2.3.14.2Information Sheet No 2/LO11 Apply Matrix

Introduction

This learning outcome covers; the determinant of a $3x3$ matrix, inverse of a $3x3$ matrix, solutions of three unknown simultaneous equations and the calculations on eigen values and eigen vectors.

Definition of key terms

Matrix: This is a set of real or complex numbers arranged in rows and columns to form a rectangular array and it is always denoted by capital letters.

Determinant: It is a physical quantity/value assigned to any square matrix. E.g. given $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - cd$

Inverse: The inverse of a matrix Q^{-1} is the matrix than when multiplied by the original matrix Q gives the identity matrix. i.e. $Q\ Q^{-1}$=I.

Content/Procedures/Methods/Illustrations

11.1 Determinant and inverse of 3x3 matrix are obtained as per the method.

A matrix with P rows and Q columns is called a $P \times Q$ matrix and is of order $P \times Q$. Square brackets [ ] or round brackets () are used when writing a matrix. For example, $\begin{bmatrix} 1 & 6 \\ 9 & 8 \end{bmatrix}$ or $\begin{pmatrix} 1 & 6 \\ 9 & 8 \end{pmatrix}$

Matrices are named by stating the number of rows followed by the number of columns. For example, $\begin{bmatrix} 1 & 9 & 7 \\ 6 & 5 & 4 \\ 2 & 3 & 6 \end{bmatrix}$ is a 3×3 matrix

. $\begin{bmatrix} 5 & 6 & 4 & 3 \\ 1 & 7 & 8 & 2 \\ 10 & 9 & 6 & 5 \end{bmatrix}$ is a 3×4 matrix

A line matrix (row matrix) consists of one row only e.g. $(5\ 6\ 4\ 3)$ while a column matrix consists of one column only e.g. $\begin{bmatrix} 5 \\ 1 \\ 10 \end{bmatrix}$

Double suffix notation

Each unit in a matrix can be defined by double suffixes. The first suffix is the row while the second is the column.

E.g. $\begin{pmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{pmatrix}$ Thus P_{32} is on the third row and the second column.

Addition and subtraction of matrices

For two or more matrices to be added or subtracted, they must be of the same order. The operation is done by adding or subtracting the corresponding elements.

Example:

$$\begin{pmatrix} 5 & 6 & 7 \\ 1 & 10 & 2 \\ 4 & 3 & 8 \end{pmatrix} + \begin{pmatrix} 12 & 9 & 8 \\ 6 & 5 & 3 \\ 9 & 7 & 2 \end{pmatrix} = \begin{pmatrix} 17 & 15 & 15 \\ 7 & 16 & 5 \\ 13 & 10 & 10 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 5 & 1 \\ 2 & 7 & 8 \\ 3 & 6 & 9 \end{pmatrix} - \begin{pmatrix} 3 & 6 & 9 \\ 5 & 5 & 4 \\ 4 & 12 & 8 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -8 \\ -3 & 2 & 4 \\ -1 & -6 & 1 \end{pmatrix}$$

Transposing a matrix

Example

$$\mathbf{Q} = \begin{pmatrix} 3 & 6 & 9 \\ 5 & 5 & 4 \\ 4 & 12 & 8 \end{pmatrix}, \qquad Q^T = \begin{pmatrix} 3 & 5 & 4 \\ 6 & 5 & 12 \\ 9 & 4 & 8 \end{pmatrix}$$

Determinant of a 3x3 matrix

Suppose Q is a 3×3 matrix, the determinant of Q denoted by det Q or $|Q|$ is obtained by

$$\begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{pmatrix}$$

$$\det Q\ or |Q| = q_{11} \begin{vmatrix} q_{22} & q_{23} \\ q_{32} & q_{33} \end{vmatrix} - q_{12} \begin{vmatrix} q_{21} & q_{23} \\ q_{31} & q_{33} \end{vmatrix} + q_{13} \begin{vmatrix} q_{21} & q_{22} \\ q_{31} & q_{32} \end{vmatrix}$$

$$det\ Q = q_{11}(q_{22}q_{33} - q_{23}q_{32}) - q_{12}(q_{21}q_{33} - q_{31}q_{23}) + q_{13}(q_{21}q_{32} - q_{31}q_{22})$$

$$det\ Q = q_{11}q_{22}q_{33} - q_{11}q_{23}q_{32} - q_{12}q_{21}q_{33} + q_{12}q_{31}q_{23} + q_{13}q_{21}q_{32} - q_{13}q_{31}q_{22}$$

$$det\ Q = q_{11}q_{22}q_{33} + q_{12}q_{31}q_{23} + q_{13}q_{21}q_{32} - q_{13}q_{31}q_{22} - -q_{11}q_{23}q_{32} - q_{12}q_{21}q_{33}$$

Thus

det Q= $\begin{vmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{vmatrix}\begin{vmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{vmatrix}$

Example

Q=$\begin{pmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 1 \end{pmatrix}$

$$det\ Q = (2\times6\times1)+(3\times4\times8)+(4\times5\times9)-(2\times7\times9)-(3\times5\times1)-(4\times6\times8)$$

$$det\ Q = 27$$

Minors and cofactors of a 3x3 matrix

Suppose R is a 3×3 matrix, R=$r_{ij}$and S is 3×3 matrix obtained from R by deleting its i^{th} row and j^{th} column. Then the determinant of s_{ij} is called the minor of the element r_{ij} of R.

Cofactors are obtained by multiplying $(-1)^{i+j}$ by the submatrix of the matrix.

Example.

If T=$\begin{pmatrix} 5 & 3 & 8 \\ 1 & 10 & 2 \\ 4 & 6 & 7 \end{pmatrix}$ Minor of T =$\begin{pmatrix} 58 & -1 & -34 \\ -27 & 3 & 18 \\ -74 & 2 & 47 \end{pmatrix}$

Cofactors of T=$\begin{pmatrix} 58 & 1 & -34 \\ 27 & 3 & -18 \\ -74 & -2 & 47 \end{pmatrix}$

Adjoint of a 3x3 matrix

Adjoint is the transpose of the cofactors of a matrix denoted by $Adj\ T$.

$Thus$

Adj T= $\begin{pmatrix} 58 & 27 & -74 \\ 1 & 3 & -2 \\ -34 & -18 & 47 \end{pmatrix}$

Inverse of a 3x3 matrix

The inverse of T, i.e. T^{-1} is given by;

. $T^{-1} = \frac{adjT}{Det\ T}$

$\det T = 5(58) - 3(-1) + 8(-34) = 21$

T^{-1}=$\frac{1}{21}\begin{pmatrix} 58 & 27 & -74 \\ 1 & 3 & -2 \\ -34 & -18 & 47 \end{pmatrix}$=$\begin{pmatrix} \frac{58}{21} & \frac{27}{21} & \frac{-74}{21} \\ \frac{1}{21} & \frac{3}{21} & \frac{-2}{21} \\ \frac{-34}{21} & \frac{-18}{21} & \frac{47}{21} \end{pmatrix}$

11.2 Solutions of simultaneous equations in three unknowns are obtained as per the procedure

Consider 3 sets of linear equations

$r_{11}x + r_{12}y + r_{13}z = C_1$

$r_{21}x + r_{22}y + r_{23}z = C_2$

$r_{31}x + r_{32}y + r_{33}z = C_3$

Then the values x, y and Z are obtained by,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}^{-1} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix}$$

Example

Obtain the values $of\ p, q, r$ in the following system of simultaneous equations.

$3p + q + 2r = 5$

$5p + 3q + 2r = 7$

$9p + 8q + 7r = 3$

Solution

$$\begin{pmatrix} 3 & 1 & 2 \\ 5 & 3 & 2 \\ 9 & 8 & 7 \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 3 & 1 & 2 \\ 5 & 3 & 2 \\ 9 & 8 & 7 \end{pmatrix}^{-1} \begin{pmatrix} 5 \\ 7 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 1 & 2 \\ 5 & 3 & 2 \\ 9 & 8 & 7 \end{pmatrix}^{-1} = \frac{1}{24} \begin{pmatrix} 5 & 9 & -4 \\ -17 & 3 & 4 \\ 13 & -15 & 4 \end{pmatrix} = \begin{pmatrix} \frac{5}{24} & \frac{3}{8} & \frac{-1}{6} \\ \frac{-17}{24} & \frac{1}{8} & \frac{1}{6} \\ \frac{13}{24} & \frac{-5}{8} & \frac{1}{6} \end{pmatrix}$$

Thus $\begin{pmatrix} p \\ q = \\ r \end{pmatrix} \begin{pmatrix} \frac{19}{6} \\ \frac{-13}{6} \\ \frac{-7}{6} \end{pmatrix}$

This implies that $p = \frac{19}{6}, q = \frac{-13}{6}, r = \frac{-7}{6}$

11.3 Calculation involving Eigen values and Eigen vectors are performed

If Q is a $m \times m$ matrix over some field N, then $\omega \varepsilon N$ is an eigen value of Q if some non-zero vector (column) rεN^m then $Qr = \omega r$. Thus, r is an eigen vector of Q which belong to the eigen value w.

Example

Find the eigen values and the eigen

Vectors associated with the matrix.

Q= $\begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{pmatrix}$

Solution

ω is an eigen value of Q is $|\omega I - Q|$=0 where I is the identity matrix.

$$|\omega I - Q| = \left| \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix} - \begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{pmatrix} \right| = 0$$

$$\begin{vmatrix} \omega - 3 & -1 & 0 \\ 0 & \omega - 1 & 1 \\ 0 & -2 & \omega - 4 \end{vmatrix} = 0$$

$$\omega - 3 \begin{vmatrix} \omega - 1 & 1 \\ -2 & \omega - 4 \end{vmatrix} = 0$$

$$\omega - 3[(\omega - 1)(\omega - 4) + 2\,] = 0$$

$$\omega - 3[(\omega - 2)(\omega - 3)] = 0 \qquad \omega = 3 \; or \; \omega = 2$$

Thus, the eigen values are $\omega = 3, 3 \; and \; \omega = 2.$

For eigen vectors, solve the equation $|\omega I - Q|$ r=0

When $\omega = 2$

$|\omega I - Q|$ r=0

$$\begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & -2 \end{pmatrix} \begin{pmatrix} p \\ s \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$-p - s = 0$

$S + t = 0$

$-2s - 2t = 0 \qquad s = 0 \qquad t = 0 \qquad p = 0$

Example:

Find the eigen values and the eigen vectors associated with.

M= $\begin{pmatrix} 7 & 0 & 0 \\ 6 & 3 & 1 \\ 2 & 10 & 6 \end{pmatrix}$

Solution

Suppose ω is an eigen value of M, then $|\omega I - M|$=0

$$|\omega I - M| = \left| \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix} - \begin{pmatrix} 7 & 0 & 0 \\ 6 & 3 & 1 \\ 2 & 10 & 6 \end{pmatrix} \right| = 0$$

$$= \begin{vmatrix} \omega - 7 & 0 & 0 \\ -6 & \omega - 3 & -1 \\ -2 & -10 & \omega - 6 \end{vmatrix} = 0$$

$$\omega - 7 \begin{vmatrix} \omega - 3 & -1 \\ -10 & \omega - 6 \end{vmatrix}$$

$\omega - 7[(\omega - 3)(\omega - 6) - 5] = 0$
$\omega - 7[\omega^2 - 6\omega - 3\omega + 18 - 10] = 0$
$\omega - 7[\omega^2 - 9\omega + 8] = 0$
$(\omega - 7)(\omega - 1)(\omega - 8) = 0$
$\omega = 7,1 \; or \; 8$

Thus, the eigen values are 1, 7 and 8

For eigen vectors;

Take $|\omega I - Q|r = 0$

$$\begin{pmatrix} \omega - 7 & 0 & 0 \\ -6 & \omega - 3 & -1 \\ -2 & -10 & \omega - 6 \end{pmatrix} \begin{matrix} x \\ y \\ z \end{matrix} = 0$$

When $\omega = 1$,

We have $\begin{pmatrix} -6 & 0 & - \\ -6 & -2 & -1 \\ -2 & 10 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$

$-6x = 0 \quad thus \; x = 0$
$-6y - 2y - z = 0$
$-2x - 10y - 5z = 0$
$-2y - z = 0$
$-10y - 5z = 0$
$-10y - 10z = 0$
$-10y - 5z = 0$
$-5z = 0$
$Z = 0 \qquad y = 0$

When $\omega = 7$

We have $\begin{pmatrix} 0 & 0 & 0 \\ -6 & 4 & -1 \\ -2 & -10 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$

$x = 0, \quad y = 0 \quad z = 0$

When $\omega = 8$

$$\begin{pmatrix} 1 & 0 & 0 \\ -6 & 5 & -1 \\ -2 & -10 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$x = 0 \qquad -6x + 5y - z = 0$
$-2x - 10y + 2z = 0$
$5y - z = 0$
$-10y + 2z = 0 \qquad y = z = 0$

Conclusion

This learning outcome covered the determinant of a $3x3$ matrix, solutions of three unknows simultaneous equation and the calculations Eigen values and Eigen vectors.

Further Reading

Read more on

1. Multiplication of two or more matrices. Matrix algebra useful for statistics by John Wiley and sons.
2. Eigen values and eigen values. Iterative methods for computing eigen values and eigen vectors by Panju M.

2.3.14.3Self-Assessment

Written Assessment

1.Find the determinant of the following matrix.

$$\begin{pmatrix} 4 & 3 & -1 \\ -17 & 7 & 9 \\ 13 & 25 & 1 \end{pmatrix}$$

a) 46
b) 48
c) 52
d) 64

2.Find the determinant of the following matrix.

$$\begin{pmatrix} 10 & 5 & 1 \\ 3 & 6 & 2 \\ 4 & 7 & 9 \end{pmatrix}$$

a) 300
b) 256
c) 272
d) 302

3.Find the inverse of the following matrices.

$$\begin{pmatrix} 3 & -5 & -2 \\ 2 & -2 & 4 \\ -3 & 8 & -5 \end{pmatrix}$$

a) $\begin{pmatrix} \frac{11}{38} & \frac{41}{76} & \frac{6}{19} \\ \frac{1}{38} & \frac{21}{76} & \frac{4}{19} \\ \frac{-5}{38} & \frac{9}{46} & \frac{-1}{19} \end{pmatrix}$

b) $\begin{pmatrix} \frac{76}{38} & \frac{65}{38} & \frac{11}{38} \\ \frac{21}{38} & \frac{23}{76} & \frac{9}{19} \\ \frac{-10}{19} & \frac{9}{76} & \frac{-1}{19} \end{pmatrix}$

c) $\begin{pmatrix} \frac{3}{38} & \frac{-5}{38} & \frac{-1}{19} \\ \frac{1}{19} & \frac{-1}{19} & \frac{2}{19} \\ \frac{-3}{38} & \frac{4}{9} & \frac{-5}{38} \end{pmatrix}$

d) $\begin{pmatrix} \frac{11}{19} & \frac{82}{76} & \frac{12}{19} \\ \frac{1}{19} & \frac{42}{76} & \frac{8}{19} \\ \frac{-5}{19} & \frac{9}{76} & \frac{-2}{19} \end{pmatrix}$

4.Find the inverse of the following matrix

$$\begin{pmatrix} 13 & 12 & 9 \\ 2 & 1 & 3 \\ 2 & -10 & 8 \end{pmatrix}$$

a) $\begin{pmatrix} \frac{19}{88} & \frac{-93}{88} & \frac{27}{176} \\ \frac{-5}{88} & \frac{43}{88} & \frac{-21}{176} \\ \frac{-1}{88} & \frac{7}{8} & \frac{-1}{16} \end{pmatrix}$

b) $\begin{pmatrix} \frac{11}{88} & \frac{-93}{88} & \frac{27}{176} \\ \frac{-3}{88} & \frac{43}{88} & \frac{-23}{176} \\ \frac{-1}{4} & \frac{7}{8} & \frac{-3}{16} \end{pmatrix}$

c) $\begin{pmatrix} \frac{21}{88} & \frac{-95}{88} & \frac{27}{176} \\ \frac{-3}{88} & \frac{47}{88} & \frac{-27}{176} \\ \frac{-5}{88} & \frac{7}{8} & \frac{-3}{16} \end{pmatrix}$

d) $\begin{pmatrix} \frac{3}{4} & \frac{-1}{6} & \frac{-2}{3} \\ \frac{1}{5} & \frac{1}{7} & \frac{2}{9} \\ \frac{3}{7} & \frac{5}{7} & 1 \end{pmatrix}$

5.Obtain the values p, q and r in the following equations.

$3p - 2r + 6q = 7$

$-7p + 4r - 8q = 15$

$9p - 3r + 17q = 3$

a) P=$\frac{31}{13}$ q= $\frac{15}{26}$ $r = \frac{23}{26}$
b) P=$\frac{-1}{3}$ $q = \frac{5}{26}$ $r = \frac{23}{26}$
c) P=$\frac{-131}{13}$ $q = \frac{-115}{26}$ $r = \frac{123}{26}$
d) P=$\frac{131}{26}$ $q = \frac{5}{16}$ $r = \frac{17}{19}$

6.Obtain the values of p, q and r in the following equation

$2p + 7r + 18q = 0$

$15p + 19r + 20q = 3$

$17p - 5r - 7q = -10$

a) $P = \frac{7}{8}$ $q = \frac{9}{10}$ $r = \frac{17}{823}$
b) $P = \frac{-1897}{4115}$ $q = \frac{652}{823}$ $r = \frac{-1057}{4115}$
c) $p = \frac{97}{115}$ $q = \frac{52}{23}$ $r = \frac{57}{15}$
d) $p = \frac{5}{6}$ $q = \frac{2}{3}$ $r = \frac{1}{5}$

7. Obtain the eigen values of $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix}$
   a) 3, 3, 4
   b) 2, 2, 3
   c) 2, 2, 2
   d) 1, 2, 3
8. Evaluate how to obtain the determinant of matrix.
9. Compare between adjoint and inverse of a matrix.
10. Differentiate between minors and cofactors?

11. Justify whether an eigen value the same as an eigen vector.

12. Solve the following system of simultaneous equations.

$3p + q + 2r = 5$

$5p + 3q + 2r = 7$

$9p + 8q + 7r = 3$

Oral Assessment

1. Define an inverse of a matrix.
2. Discuss the process of finding the eigen vector.

Practical Assessment

In a given department, 3 lecturers may teach 3 units per week. The number of hours required on each unit to be examined are given in the table below. The department chairperson would like to assign the lecturers the units so that the number of hours is minimized. Find the specific unit that each lecturer should be assigned.

Lecturer	Math’s	Science	English
P	114	116	112
Q	19	228	109
R	113	600	110

2.3.14.4 Tools, Equipment, Supplies and Materials

- Scientific calculator
- 15 cm ruler
- Pen, pencil

2.3.14.5 References

Aitken, A.C (2017) Determinant and matrices Read Book Ltd.

Panju,M(2011) Iterative methods for computing eigen values and eigen vectors ar.xiv 11051185

Searle. S.R., & Khun, A.I. (2017) matrix algebra useful for statistics. John wiley & sons.